Nossa Senhora dos Mártires

Analysis of the curves:
Floors and Futtocks

Dr. Thomas Vogel
College Station, Fall 2000
Introduction

The analysis of the futtocks was certainly the most frustrating task in the tentative reconstruction of Mártires' hull. Not only do the arcs do not seem to be constant, but they do not even seem to follow a clear rule as their radii decrease towards the bow.

There seems to be plenty of evidence for the construction of ships through the whole-molding system using futtocks with just one circular arc. This system was much easier from the point of view of the timber suppliers, who could go to the woods with only one template and fell all the trees necessary to fulfill a certain request, but above all it was much easier for the contractor, who could handle the timbers with much more freedom, in terms of storing and moving around in the shipyard. It is easy to imagine that if each frame of a certain vessel had to be built with a particular arc for each particular station all futtocks would have to be numbered or clustered, and stored accordingly, and the management of the stocks would have been much more difficult.

Many authors mention a fairly simple and widely spread method used to vary the breadth of the frames according to the needs of the shipwright along the sequences of frames before and abaft the master frame. This method was already implied in Matthew Baker's Fragments of Ancient Shipwrightry and in Fernando Oliveira's Livro da Fabrica das Naus, and is still in use in Mediterranean traditional shipyards. It consists mainly in sliding the mold of the futtock down over the mold of the floor in order to open the breath of the frame at its upper end (FIG. 1). Another way to achieve this consists of rotating the mold of the futtock around the turn of the bilge.

If this widening of the upper lip of the first futtocks is not performed, the narrowing of the bottom is reflected upwards, generally narrowing the upper deck more than required. For this reason it was common to draw the main deck before assembling the frames, and to fix the breadth at several levels for each station to determine the overture of the first futtocks that would guarantee these pre-determined measures.
FIG. 1 - Whole molding after Kostas Danisiadis "Methods used to Control the Form of the Vessels in the Greek Traditional Boatyards". Drawing: Filipe Castro.

This practice is probably much older than the whole molding tradition, and is already documented by Zorzi Timbotta.

Both Fernando Oliveira and Matthew Baker mention this technique, the first by referring to the drawing of the main deck, and the second by showing three curves on one of his best known diagrams, being the first for the rising of the bottom, the second for its narrowing, and the third for the narrowing of the weather deck.

However, the puzzling aspect of Oliveira's work is that in one of his drawings he defines a system that implies a different radius for each futtock (FIG. 2).

I had always thought that this was a typical misunderstanding of Father Oliveira, who knew very much about shipbuilding but had never built a ship. However, the futtocks of the Pepper Wreck do decrease their radii as they move away from the master frame, even considering the odd results obtained for futtock B4E. It seems that in this section of the vessel, two sizes of futtocks were used, the ones with larger radii closer to the midships frame, and those with smaller radii towards the bow. Although this practice is not documented to my knowledge anywhere except in Oliveira's Liuro da Fabrica das Naves, it seems possible that the frames were assembled with futtocks with radii that vary gradually towards the extremities by steps rather than continuously, as Oliveira seems to suggest. This solution would be a compromise between his system, which suggests a different radius for each futtock in all of the pre-designed frames, and the normal way, which implies the use of different extensions of a set of futtocks with the same radii. It is unfortunate that insufficient data exists for this wreck, precluding a clear understanding of the method used to cut the first futtocks of this vessel.

Only six futtocks were preserved to some extent, and no futtock was preserved in its entirety, many of the timbers presented large gaps in their sections and some had patches filling the larger gaps, showing the difficult situation the shipwrights had to face when building large vessels with a small stock of suitable timber. For all these reasons, it is quite daring to try to establish theories from the available data.

When analyzed, the futtock curvatures do not match any of the expectations. Although they clearly show to have a turn of the bilge arc and a futtock arc, the radii found through the several methods of analysis utilized did not follow any particular pattern, at least when the typical way in which these timbers are believed to have been cut in the period under consideration.
Instead of a regular design obtained by the application of a standard mold that would slide or tilt to allow the shipwright to obtain a fair set of longitudinal runs while rising and narrowing the bottom of the vessel, it seems that the futtock’s radii diminish towards the bow, as Oliveira indicates in his book. The irregularity of the values obtained is far from the unlikely precision implied in Oliveira’s drawing. This scheme has often been interpreted as the result of a poor understanding of the building technique, because it seems totally unfeasible to cut each pair of futtocks with a particular radius, for all the 39 pre-designed frames required. One can hardly imagine the problems this practice would raise with regard to provisioning, stocking and handling of compass wood in any shipyard, should this process be applied with the precision implied in Oliveira’s drawing.

However, when we look at the values obtained from the six futtocks that were analyzed, there seems to be a trend towards the reduction of the futtock’s arc as we move away from the master frames. The irregularity of the values obtained suggests that the frames were assembled on the ground – as the fastening pattern between floors and futtocks shows without any doubt – over a template were the maximum beam and the beam on the main deck were marked with precision for each station, and the futtocks were chosen “by eye” from the pile of timber stock. The larger radii were naturally utilized on the frames that are closer to the midships, and the smaller radii in the extremities.

A pile of timber probably buried in the sand after the 1755 earthquake was found in 1996 on the site of Lisbon’s 16th and 17th century shipyard, by a contractor who was building a large underground park on Praça do Município. Still under study, it encompasses a few keel sections with the rabbets already opened, stored next to a number of roughly cut flat, “V”, and “Y” shaped timbers, obviously meant for the construction of floor timbers. No curved timbers were found anywhere nearby, perhaps suggesting that these were stored separately, and reinforcing the idea that the futtocks were chosen from a pile of more or less equally curved timbers, the bigger radii used to assemble wider frames, the smaller radii for the frames to be placed closer to the extremities.
Methodology

To find the values of the arcs of these six futtocks, three different methods were used. The first, a very simple geometric method, consisted of the graphic resolution of the center of the circle for a given group of three points (FIG. 3). This technique was quickly dismissed after a small number of tests, since the values obtained for the radius varied largely with the points chosen. These tests were made on 1:10 scale drawings obtained by reducing the full scale tracings using a computer-aided design program, and it is possible that any errors made during tracing were compounded during the scanning process.

FIG. 3 - Graphic method used to find the center of curvature of the futtocks.
Drawing: Filipe Castro.

The second method consisted of the use of templates with circular curves which were overlaid on the drawings of the futtocks to determine which curve fit best over each futtock. This method was also not very accurate, as the average preserved of only three meters allowed a very wide number of arcs to fit equally well over the outline of each futtock.

The third method consisted of a mathematical analysis of the lists of coordinates \((x, y)\) that define each one of the lower surfaces of the futtocks at 10 cm intervals. This
analysis was performed with the help of two computer programs developed by Dr. Thomas Vogel of Texas A&M University Mathematics Department, which run on a Maple $\beta^3$ environment. The first of these programs finds the best fit circular curve for a given number of points. The second program finds the radius of each three consecutive points and lists the series of radii obtained (See Appendix A).

The available data was run through the two programs with rather inconclusive results. The radii obtained using the first program clearly showed the existence of a turn of the bilge arc and a futtock arc, and suggested the existence of a tumblehome arc in the longer futtocks. However, the results varied too widely to allow for any further analysis, and thus the second program was employed. Convinced that the $x$ and $y$ values taken at the extremities of the futtocks might create some form of noise in the computation of the values of the radii if there were three arcs, several combinations of points were tested. As a result, the radius of the best fit circular curve for each futtock was determined, for nine different combinations of points: for the whole extension of the futtocks every 10 cm and every 30 cm, and excluding 50 cm in each one of the extremities, for five points, and for five different combinations of three points. I have built a table with all the values given by the computer program (Table I).

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<tr>
<th>Futtock</th>
<th># 10 cm</th>
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</table>

(1) and (2) calculated for the whole length; (3) to (7) for the central portion of the futtock in the following way: (4) $x=50/150/250$; (5) $x=60/160/260$; (6) $x=40/140/240$; (7) $x=40/150/260$; and (8) $x=60/130/230$.

The computerized values were then compared with the ones obtained from the templates (Table II), using the two columns of Table I that seem most reliable or relevant.
to this analysis; the values obtained for the central portion considering five points along
the curve; and the average of the eight values.

<table>
<thead>
<tr>
<th>Futtocks</th>
<th>Templates</th>
<th>Central Portion (5 points)</th>
<th>Average of the best-fit curves</th>
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It has been suggested that these frames were pre-designed and pre-assembled
according to the rules expressed by the Portuguese treatises and texts of shipbuilding of
their time. That they were in fact pre-assembled is obvious. All spikes were clenched on
the side of the futtocks and encased in recessed cavities. Their heads were encased in
countersink holes on the floors, at the side of the heads, and grooves were adzed on the
futtocks to house the clenched points of the spikes. The floors leaned against the futtocks
of the previous frame without any space between them, and this arrangement would have
been impossible if the floors were not already attached to the futtocks.

<table>
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<th>Author</th>
<th>Master Frames</th>
<th>Pre-designed frames</th>
<th>Bibliographic ref.</th>
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<td>15 before; 18 abaft</td>
<td>Fernando Oliveira: 94 and 174</td>
</tr>
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</tr>
<tr>
<td>Sebastiao Thernudo</td>
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<td>5 before; 5 abaft</td>
<td>Lavanha: 115 and 228</td>
</tr>
<tr>
<td>Goncalo Ruix</td>
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<td>15 before; 15 abaft</td>
<td>Lavanha: 117 and 240</td>
</tr>
<tr>
<td>João Baptista Lavanha</td>
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<td>5 before; 5 abaft</td>
<td>Lavanha: 157 and 163, Fl. 72</td>
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<td>Manoel Fernandez</td>
<td>3</td>
<td>15 before; 15 abaft</td>
<td>Fernandez: fl. 1 v.6</td>
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</table>

As many pre-designed floors as the number of ramos in the keel.

The evidence for the pre-design of the frames is redundant, since it is impossible
to cut and assemble them without designing them first. There are some clues, and many
doubts as to the way in which they were pre-designed. Different authors present or suggest different ways to design the frames to obtain a good hull shape, with fair runs and smooth ends, that could cut through the water without plunging the bow, and steer efficiently without sinking the stern (Table III).

Other authors indicate different solutions. In Spain, Diego Garcia de Palacio mentions nine pre-designed floors to the bow and six to the stern, and Tomé Cano prescribes fourteen pre-designed floors on each side of the master frame.

The clues left on the structure of the Pepper Wreck are scarce yet speak volumes. Although only eleven floors were partially preserved, and of these only a few were not badly broken. All the floors that were not broken were lost during the period between 1997 and 1999, due to lack of means to expose and protect the structure after it was covered by several meters of sand during a series of storms in the winter of 1997.

Profiles recorded in 1996 and 1997 using an electronic goniometer revealed the position of the planking, rather than the curves of the lower face of the floors, and are not very useful for the analysis of the shape of the frames. Moreover, many construction marks could not have been observed in situ, as the timbers required thorough cleaning before most of the marks could be exposed.

However, the available evidence allows a tentative reconstruction of the process followed by the shipwrights of the Pepper Wreck. The most important clues for the understanding of the construction process are the numbers engraved on the floors, the rate at which the height of the floors grows over the keel, the stumps marks which suggest the position of the turn of the bilge, and the curves of the futtocks.
Discussion

The study of the curvature of these futtocks was difficult and somewhat inconclusive. Not only was the construction generally crude, with no smooth surfaces on any of the preserved futtocks, but the lower faces of the futtocks had undergone heavy dubbing during the construction of the vessel in order to bevel the outer face of the frames to receive the planking. The small number and poor preservation of the extant futtocks made their analysis even more difficult.

An attempt was made to determine the arcs of the futtocks with the best possible accuracy, resulting in very interesting, but puzzling results. The methodology adopted in the analysis of the futtocks was largely determined by the assumptions that the futtocks had one single arc, as indicated by Oliveira in his drawings and that all the futtocks had the same radius. Both of these assumptions were proven to be faulty. Maybe in the near future we will come across better evidence and be able to state with a fair degree of certainty how these frames were designed.
Appendix A

Data

College Station, March 2001
Futlocks' curves

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<tr>
<th>Futlock</th>
<th># 10 cm</th>
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50/150/250 60/160/260 40/140/240 40/150/260 60/150/240

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C2
Starboard
Floor X
This worksheet is to find a circle which best fits a list of data points. The \( x \) coordinates are given as a list \( xx \), and the corresponding \( y \) coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, "Buttock VIII, B3E".

\[
\text{restart:}
\]

\[
xx := [12.5, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 156];
\]

\[
xx := [12.5, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 156];
\]

\[
yy := [6.5, 10, 13, 15.5, 17.5, 19.5, 21.5, 23, 25, 26.5, 28, 29.5, 31, 32, 34.5];
\]

\[
n := 15;
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \((x_i, y_i)\), \(i = 1 \ldots n\), then the \( n \) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0,\) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x0, y0, R) \rightarrow \sum_{i=1}^n ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

\[
eq 1 := \text{diff}(F(x0, y0, R), x0) = 0;
\]

\[
eq 1 := 2 ((12.5 - x0)^2 + (6.5 - y0)^2 - R^2) (-40 + 2 x0)
\]

\[
+ 2 ((20 - x0)^2 + (10 - y0)^2 - R^2) (-60 + 2 x0)
\]

\[
+ 2 ((30 - x0)^2 + (13 - y0)^2 - R^2) (-80 + 2 x0)
\]

\[
+ 2 ((40 - x0)^2 + (15.5 - y0)^2 - R^2) (-100 + 2 x0)
\]

\[
+ 2 ((50 - x0)^2 + (17.5 - y0)^2 - R^2) (-120 + 2 x0)
\]

\[
+ 2 ((60 - x0)^2 + (19.5 - y0)^2 - R^2) (-140 + 2 x0)
\]

\[
+ 2 ((70 - x0)^2 + (21.5 - y0)^2 - R^2) (-160 + 2 x0)
\]

\[
+ 2 ((80 - x0)^2 + (23 - y0)^2 - R^2) (-180 + 2 x0)
\]

\[
+ 2 ((90 - x0)^2 + (25 - y0)^2 - R^2) (-200 + 2 x0)
\]

\[
+ 2 ((100 - x0)^2 + (26.5 - y0)^2 - R^2) (-220 + 2 x0)
\]

\[
+ 2 ((110 - x0)^2 + (28 - y0)^2 - R^2) (-240 + 2 x0)
\]

\[
+ 2 ((120 - x0)^2 + (29.5 - y0)^2 - R^2) (-260 + 2 x0)
\]

\[
+ 2 ((130 - x0)^2 + (31 - y0)^2 - R^2) (-280 + 2 x0)
\]

\[
+ 2 ((140 - x0)^2 + (32 - y0)^2 - R^2) (-312 + 2 x0) = 0
\]

\[
eq 2 := \text{diff}(F(x0, y0, R), y0) = 0;
\]
\[eq2 := 2 \left( (12.5 - x_0)^2 + (6.5 - y_0)^2 - R_0^2 \right) \left( -13 + 2 y_0 \right)
+ 2 \left( (20 - x_0)^2 + (10 - y_0)^2 - R_0^2 \right) \left( -20 + 2 y_0 \right)
+ 2 \left( (30 - x_0)^2 + (13 - y_0)^2 - R_0^2 \right) \left( -26 + 2 y_0 \right)
+ 2 \left( (40 - x_0)^2 + (15.5 - y_0)^2 - R_0^2 \right) \left( -31.0 + 2 y_0 \right)
+ 2 \left( (50 - x_0)^2 + (17.5 - y_0)^2 - R_0^2 \right) \left( -35.0 + 2 y_0 \right)
+ 2 \left( (60 - x_0)^2 + (19.5 - y_0)^2 - R_0^2 \right) \left( -39.0 + 2 y_0 \right)
+ 2 \left( (70 - x_0)^2 + (21.5 - y_0)^2 - R_0^2 \right) \left( -43.0 + 2 y_0 \right)
+ 2 \left( (80 - x_0)^2 + (23 - y_0)^2 - R_0^2 \right) \left( -46 + 2 y_0 \right)
+ 2 \left( (90 - x_0)^2 + (25 - y_0)^2 - R_0^2 \right) \left( -50 + 2 y_0 \right)
+ 2 \left( (100 - x_0)^2 + (26.5 - y_0)^2 - R_0^2 \right) \left( -54 + 2 y_0 \right)
+ 2 \left( (110 - x_0)^2 + (28 - y_0)^2 - R_0^2 \right) \left( -56 + 2 y_0 \right)
+ 2 \left( (120 - x_0)^2 + (29.5 - y_0)^2 - R_0^2 \right) \left( -59 + 2 y_0 \right)
+ 2 \left( (130 - x_0)^2 + (31 - y_0)^2 - R_0^2 \right) \left( -62 + 2 y_0 \right)
+ 2 \left( (140 - x_0)^2 + (32 - y_0)^2 - R_0^2 \right) \left( -64 + 2 y_0 \right)
+ 2 \left( (156 - x_0)^2 + (34.5 - y_0)^2 - R_0^2 \right) \left( -69 + 2 y_0 \right) = 0\]

\[eq3 := \text{diff} \left( \text{diff} \left( \text{R} \left( x_0, y_0, R \right), R \right), R \right) = 0;\]

\[eq3 := -4 \left( (12.5 - x_0)^2 + (6.5 - y_0)^2 - R_0^2 \right) R - 4 \left( (20 - x_0)^2 + (10 - y_0)^2 - R_0^2 \right) R
- 4 \left( (30 - x_0)^2 + (13 - y_0)^2 - R_0^2 \right) R - 4 \left( (40 - x_0)^2 + (15.5 - y_0)^2 - R_0^2 \right) R
- 4 \left( (50 - x_0)^2 + (17.5 - y_0)^2 - R_0^2 \right) R - 4 \left( (60 - x_0)^2 + (19.5 - y_0)^2 - R_0^2 \right) R
- 4 \left( (70 - x_0)^2 + (21.5 - y_0)^2 - R_0^2 \right) R - 4 \left( (80 - x_0)^2 + (23 - y_0)^2 - R_0^2 \right) R
- 4 \left( (90 - x_0)^2 + (25 - y_0)^2 - R_0^2 \right) R - 4 \left( (100 - x_0)^2 + (26.5 - y_0)^2 - R_0^2 \right) R
- 4 \left( (110 - x_0)^2 + (28 - y_0)^2 - R_0^2 \right) R - 4 \left( (120 - x_0)^2 + (29.5 - y_0)^2 - R_0^2 \right) R
- 4 \left( (130 - x_0)^2 + (31 - y_0)^2 - R_0^2 \right) R - 4 \left( (140 - x_0)^2 + (32 - y_0)^2 - R_0^2 \right) R
- 4 \left( (156 - x_0)^2 + (34.5 - y_0)^2 - R_0^2 \right) R = 0\]

\[\text{sol} := \text{solve} \left( \{eq1, eq2, eq3\}, \{x_0, y_0, R\} \right);\]

\[\text{sol} := \{ R = 0, x_0 = 80.08476623 - 75.54880189 \times 1 - 22.61639195 - 13.94450520 \times 1 \},\]

\[R = 0, x_0 = 80.08476623 + 75.54880189 \times 1 + 22.61639195 + 13.94450520 \times 1 \},\]

\[R = 0, x_0 = 81.65305983 + 8.080052468 \times 1 - 22.90449240 + 43.67784752 \times 1 \},\]

\[R = 0, x_0 = 81.65305983 - 8.080052468 \times 1, y_0 = 22.90449240 + 43.67784752 \times 1 \},\]

\[x_0 = 207.6328483, y_0 = -659.2509552, R = -694.6144790 \}.
\]

\[x_0 = 207.6328483, y_0 = -659.2509552, R = 694.6144790 \}.
\]

\[\text{circ} := (x - x_0)^2 + (y - y_0)^2 = R^2\]
> with(plots);
[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
    contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
    fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
    listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
    pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
    polyhedrplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
    sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]
> p1 := implicitplot(subs(solve(7, circl), x=0..400, y=-300..300, numpoints=4000):
> with(stats[statplots]);
[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift,
   yzexchange, zscale, zshift]
> p2 := scatterplot(xx, yy, color=black):
> display([p1, p2], scaling=constrained);
C3
Starboard
Floor VIII
This worksheet is to find a circle which best fits a list of data points. The \( x \) coordinates are given as a list \( xx \), and the corresponding \( y \) coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, "Futlock VIII, B3E".

\[ \text{restart;} \]

\[ xx := [12.5, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180]; \]

\[ xx := [12.5, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180] \]


\[ n := 18; \]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \((x_i, y_i)\), \(i = 1 \ldots n\), then the \( n \) quantities \((x - x_0)^2 + (y - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \text{ and } R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[ F := (x_0, y_0, R) \rightarrow \text{sum} \left\{ \left( (x_i - x_0)^2 + (y_i - y_0)^2 - R^2 \right)^2, i = 1 \ldots n \right\}; \]

\[ F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} \left( (x_i - x_0)^2 + (y_i - y_0)^2 - R^2 \right)^2 \]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \text{ and } R \).

\[ \text{eq1 := diff(F(x_0, y_0, R), x_0) = 0;} \]

\[ eq1 := 2 \left( (12.5 - x_0)^2 + (7.9 - y_0)^2 - R^2 \right) (-250 + 2 x_0) \]

\[ + 2 \left( (20 - x_0)^2 + (9.7 - y_0)^2 - R^2 \right) (-40 + 2 x_0) \]

\[ + 2 \left( (30 - x_0)^2 + (11.3 - y_0)^2 - R^2 \right) (-60 + 2 x_0) \]

\[ + 2 \left( (40 - x_0)^2 + (13 - y_0)^2 - R^2 \right) (-80 + 2 x_0) \]

\[ + 2 \left( (50 - x_0)^2 + (13.9 - y_0)^2 - R^2 \right) (-100 + 2 x_0) \]

\[ + 2 \left( (60 - x_0)^2 + (15.3 - y_0)^2 - R^2 \right) (-120 + 2 x_0) \]

\[ + 2 \left( (70 - x_0)^2 + (16.4 - y_0)^2 - R^2 \right) (-140 + 2 x_0) \]

\[ + 2 \left( (80 - x_0)^2 + (17.1 - y_0)^2 - R^2 \right) (-160 + 2 x_0) \]

\[ + 2 \left( (90 - x_0)^2 + (18 - y_0)^2 - R^2 \right) (-180 + 2 x_0) \]

\[ + 2 \left( (100 - x_0)^2 + (18.8 - y_0)^2 - R^2 \right) (-200 + 2 x_0) \]

\[ + 2 \left( (110 - x_0)^2 + (19.3 - y_0)^2 - R^2 \right) (-220 + 2 x_0) \]

\[ + 2 \left( (120 - x_0)^2 + (19.7 - y_0)^2 - R^2 \right) (-240 + 2 x_0) \]

\[ + 2 \left( (130 - x_0)^2 + (20.2 - y_0)^2 - R^2 \right) (-260 + 2 x_0) \]
\[ +2 \left[ (140 - x0)^2 + (20.5 - y0)^2 - R^2 \right] (-280 + 2 x0) \\
+2 \left[ (150 - x0)^2 + (21.4 - y0)^2 - R^2 \right] (-300 + 2 x0) \\
+2 \left[ (160 - x0)^2 + (22.8 - y0)^2 - R^2 \right] (-320 + 2 x0) \\
+2 \left[ (170 - x0)^2 + (25.7 - y0)^2 - R^2 \right] (-360 + 2 x0) = 0 \\
\]

\[ eq2 := \text{diff}(F(x0, y0, R), y0) = 0; \]

\[ eq2 := 2 \left[ (12.5 - x0)^2 + (7.9 - y0)^2 - R^2 \right] (-15.8 + 2 y0) \\
+2 \left[ (20 - x0)^2 + (9.7 - y0)^2 - R^2 \right] (-19.4 + 2 y0) \\
+2 \left[ (30 - x0)^2 + (11.3 - y0)^2 - R^2 \right] (-22.6 + 2 y0) \\
+2 \left[ (40 - x0)^2 + (13 - y0)^2 - R^2 \right] (-26 + 2 y0) \\
+2 \left[ (50 - x0)^2 + (13.9 - y0)^2 - R^2 \right] (-27.8 + 2 y0) \\
+2 \left[ (60 - x0)^2 + (15.3 - y0)^2 - R^2 \right] (-30.6 + 2 y0) \\
+2 \left[ (70 - x0)^2 + (16.4 - y0)^2 - R^2 \right] (-32.8 + 2 y0) \\
+2 \left[ (80 - x0)^2 + (17.1 - y0)^2 - R^2 \right] (-34.2 + 2 y0) \\
+2 \left[ (90 - x0)^2 + (18 - y0)^2 - R^2 \right] (-36 + 2 y0) \\
+2 \left[ (100 - x0)^2 + (18.8 - y0)^2 - R^2 \right] (-37.6 + 2 y0) \\
+2 \left[ (110 - x0)^2 + (19.3 - y0)^2 - R^2 \right] (-38.6 + 2 y0) \\
+2 \left[ (120 - x0)^2 + (19.7 - y0)^2 - R^2 \right] (-39.4 + 2 y0) \\
+2 \left[ (130 - x0)^2 + (20.2 - y0)^2 - R^2 \right] (-40.4 + 2 y0) \\
+2 \left[ (140 - x0)^2 + (20.5 - y0)^2 - R^2 \right] (-41.0 - 2 y0) \\
+2 \left[ (150 - x0)^2 + (21.4 - y0)^2 - R^2 \right] (-42.8 + 2 y0) \\
+2 \left[ (160 - x0)^2 + (22.8 - y0)^2 - R^2 \right] (-45.6 + 2 y0) \\
+2 \left[ (170 - x0)^2 + (25 - y0)^2 - R^2 \right] (-50 + 2 y0) \\
+2 \left[ (180 - x0)^2 + (25.7 - y0)^2 - R^2 \right] (-51.4 + 2 y0) = 0 \\
\]

\[ eq3 := \text{diff}(F(x0, y0, R), R) = 0; \]

\[ eq3 := -4 \left[ (12.5 - x0)^2 + (7.9 - y0)^2 - R^2 \right] R - 4 \left[ (20 - x0)^2 + (9.7 - y0)^2 - R^2 \right] R \\
-4 \left[ (30 - x0)^2 + (11.3 - y0)^2 - R^2 \right] R - 4 \left[ (40 - x0)^2 + (13 - y0)^2 - R^2 \right] R \\
-4 \left[ (50 - x0)^2 + (13.9 - y0)^2 - R^2 \right] R - 4 \left[ (60 - x0)^2 + (15.3 - y0)^2 - R^2 \right] R \\
-4 \left[ (70 - x0)^2 + (16.4 - y0)^2 - R^2 \right] R - 4 \left[ (80 - x0)^2 + (17.1 - y0)^2 - R^2 \right] R \\
-4 \left[ (90 - x0)^2 + (18 - y0)^2 - R^2 \right] R - 4 \left[ (100 - x0)^2 + (18.8 - y0)^2 - R^2 \right] R \\
-4 \left[ (110 - x0)^2 + (19.3 - y0)^2 - R^2 \right] R - 4 \left[ (120 - x0)^2 + (19.7 - y0)^2 - R^2 \right] R \\
-4 \left[ (130 - x0)^2 + (20.2 - y0)^2 - R^2 \right] R - 4 \left[ (140 - x0)^2 + (20.5 - y0)^2 - R^2 \right] R \\
-4 \left[ (150 - x0)^2 + (21.4 - y0)^2 - R^2 \right] R - 4 \left[ (160 - x0)^2 + (22.8 - y0)^2 - R^2 \right] R \\
-4 \left[ (170 - x0)^2 + (25 - y0)^2 - R^2 \right] R - 4 \left[ (180 - x0)^2 + (25.7 - y0)^2 - R^2 \right] R \\
\]
\[ -4 \left( (170 - x0)^2 + (25 - y0)^2 - R^2 \right) R - 4 \left( (180 - x0)^2 + (25.7 - y0)^2 - R^2 \right) R = 0 \]

> sol := solve\{eq1, eq2, eq3\}, \{x0, y0, R\};

\[ \text{sol} := \{ R = 0, y0 = 17.16179981, x0 = 95.36312265 \}, \]
\[ \{ R = 0, y0 = 17.75269219 - 8.261638258 I, x0 = 95.02674812 - 89.47629098 I \}, \]
\[ \{ R = 0, y0 = 17.75269219 + 8.261638258 I, x0 = 95.02674812 + 89.47629098 I \}, \]
\[ \{ R = 0, y0 = 17.78685049 - 51.68136717 I, x0 = 95.39946174 + 4.773406246 I \}, \]
\[ \{ R = 0, y0 = 17.78685049 + 51.68136717 I, x0 = 95.39946174 - 4.773406246 I \}, \]
\[ \{ y0 = -670.5075666, R = -692.9614161, x0 = 158.9568363 \}, \]
\[ \{ y0 = -670.5075666, x0 = 158.9568363, R = 692.9614161 \} \]

> circ := (x-x0)^2 + (y-y0)^2 = R^2;

\[ \text{circ} := (x - x0)^2 + (y - y0)^2 = R^2 \]

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra, polyhedraplot, replplot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

> p1 := implicitplot\{subs(sol[1], circ), x=0..400, y=-300..300, numpoints=4000\};

> with(stats\{statplots\});

[boxplot, histogram, scatterplot, xscale, xshift, yexchange, xexchange, yscale, yshift, yexchange, xscale, xshift]

> p2 := scatterplot\{xx, yy, color=black\};

> display([p1, p2], scaling=constrained);
B3E
Port side
Futlock VIII
Braço B3
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled.

\[ \text{restart}; \]
\[ xx := [7.2, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 316]; \]
\[ yy := [28.3, 22.1, 20.7, 17.6, 16, 13.7, 11.9, 11, 12.4, 12.8, 13, 13.5, 13.4, 14.3, 14, 15, 16, 16.6, 15.6, 17.4, 20, 22.8, 25.3, 28.4, 32, 35.5, 38.9, 42.8, 47, 51.2, 55.4, 60.2, 65.6, 71]; \]
\[ n := 33; \]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \( [x_i, y_i] \), \( i = 1 \ldots n \), then the \( n \) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \( x_0, y_0 \), and \( R \) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n}((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).

\[
eq \text{solve}: = \text{solve}(F(x_0, y_0, R) = 0); \]

\[
eq 1 := 2((7.2 - x_0)^2 + (28.3 - y_0)^2 - R^2)(-14.4 + 2x_0) + 2((10 - x_0)^2 + (22.1 - y_0)^2 - R^2)(-20 + 2x_0) + 2((20 - x_0)^2 + (20.7 - y_0)^2 - R^2)(-40 + 2x_0) + 2((30 - x_0)^2 + (17.6 - y_0)^2 - R^2)(-60 + 2x_0) + 2((40 - x_0)^2 + (16 - y_0)^2 - R^2)(-80 + 2x_0) + 2((50 - x_0)^2 + (13.7 - y_0)^2 - R^2)(-100 + 2x_0) + 2((60 - x_0)^2 + (11.9 - y_0)^2 - R^2)(-120 + 2x_0) + 2((70 - x_0)^2 + (11 - y_0)^2 - R^2)(-140 + 2x_0) + 2((80 - x_0)^2 + (12.4 - y_0)^2 - R^2)(-160 + 2x_0) + 2((90 - x_0)^2 + (12.8 - y_0)^2 - R^2)(-180 + 2x_0) + 2((100 - x_0)^2 + (13 - y_0)^2 - R^2)(-200 + 2x_0)
\]
\[ +2 ((100 - x0)^2 + (13 - y0)^2 - R^2) (-26 + 2 y0) \\
+2 ((110 - x0)^2 + (13.5 - y0)^2 - R^2) (-27.0 + 2 y0) \\
+2 ((120 - x0)^2 + (14.3 - y0)^2 - R^2) (-28.6 + 2 y0) \\
+2 ((130 - x0)^2 + (14 - y0)^2 - R^2) (-28 + 2 y0) \\
+2 ((140 - x0)^2 + (15 - y0)^2 - R^2) (-30 + 2 y0) \\
+2 ((150 - x0)^2 + (16 - y0)^2 - R^2) (-32 + 2 y0) \\
+2 ((160 - x0)^2 + (16.6 - y0)^2 - R^2) (-33.2 + 2 y0) \\
+2 ((170 - x0)^2 + (15.6 - y0)^2 - R^2) (-34.2 + 2 y0) \\
+2 ((180 - x0)^2 + (17.4 - y0)^2 - R^2) (-34.8 + 2 y0) \\
+2 ((190 - x0)^2 + (20 - y0)^2 - R^2) (-40 + 2 y0) \\
+2 ((200 - x0)^2 + (22.8 - y0)^2 - R^2) (-45.6 + 2 y0) \\
+2 ((210 - x0)^2 + (25.3 - y0)^2 - R^2) (-50.6 + 2 y0) \\
+2 ((220 - x0)^2 + (28.4 - y0)^2 - R^2) (-56.8 + 2 y0) \\
+2 ((230 - x0)^2 + (32 - y0)^2 - R^2) (-64 + 2 y0) \\
+2 ((240 - x0)^2 + (35.5 - y0)^2 - R^2) (-71.0 + 2 y0) \\
+2 ((250 - x0)^2 + (38.9 - y0)^2 - R^2) (-77.8 + 2 y0) \\
+2 ((260 - x0)^2 + (42.8 - y0)^2 - R^2) (-85.6 + 2 y0) \\
+2 ((270 - x0)^2 + (47 - y0)^2 - R^2) (-94 + 2 y0) \\
+2 ((280 - x0)^2 + (51.2 - y0)^2 - R^2) (-102.4 + 2 y0) \\
+2 ((290 - x0)^2 + (55.4 - y0)^2 - R^2) (-110.8 + 2 y0) \\
+2 ((300 - x0)^2 + (60.2 - y0)^2 - R^2) (-120.4 + 2 y0) \\
+2 ((310 - x0)^2 + (65.6 - y0)^2 - R^2) (-131.2 + 2 y0) \\
+2 ((316 - x0)^2 + (71 - y0)^2 - R^2) (-142 + 2 y0) = 0 \\
\]

\[ \text{eq3} := -4 ((17.2 - x0)^2 + (28.3 - y0)^2 - R^2) \]

\[ R - 4 ((10 - x0)^2 + (22.1 - y0)^2 - R^2) \]

\[ R - 4 ((20 - x0)^2 + (20.7 - y0)^2 - R^2) \]

\[ R - 4 ((40 - x0)^2 + (16 - y0)^2 - R^2) \]

\[ R - 4 ((60 - x0)^2 + (11.9 - y0)^2 - R^2) \]

\[ R - 4 ((80 - x0)^2 + (12.4 - y0)^2 - R^2) \]

\[ R - 4 ((100 - x0)^2 + (13 - y0)^2 - R^2) \]

\[ R - 4 ((120 - x0)^2 + (13.5 - y0)^2 - R^2) \]

\[ R - 4 ((140 - x0)^2 + (14.3 - y0)^2 - R^2) \]

\[ R - 4 ((140 - x0)^2 + (15 - y0)^2 - R^2) \]

\[ R - 4 ((160 - x0)^2 + (16.6 - y0)^2 - R^2) \]
-4 (((180 - xo)^2 + (17.4 - yo)^2 - R^2) \cdot R - 4 (((190 - xo)^2 + (20 - yo)^2 - R^2) \cdot R
-4 (((200 - xo)^2 + (22.8 - yo)^2 - R^2) \cdot R - 4 (((210 - xo)^2 + (25.3 - yo)^2 - R^2) \cdot R
-4 (((220 - xo)^2 + (28.4 - yo)^2 - R^2) \cdot R - 4 (((230 - xo)^2 + (32 - yo)^2 - R^2) \cdot R
-4 (((240 - xo)^2 + (35.5 - yo)^2 - R^2) \cdot R - 4 (((250 - xo)^2 + (38.9 - yo)^2 - R^2) \cdot R
-4 (((260 - xo)^2 + (42.8 - yo)^2 - R^2) \cdot R - 4 (((270 - xo)^2 + (47 - yo)^2 - R^2) \cdot R
-4 (((280 - xo)^2 + (51.2 - yo)^2 - R^2) \cdot R - 4 (((290 - xo)^2 + (55.4 - yo)^2 - R^2) \cdot R
-4 (((300 - xo)^2 + (60.2 - yo)^2 - R^2) \cdot R - 4 (((310 - xo)^2 + (65.6 - yo)^2 - R^2) \cdot R
-4 (((316 - xo)^2 + (71 - yo)^2 - R^2) \cdot R = 0

> sol := solve({eq1, eq2, eq3}, {xo, yo, R});

sol := \{xo = 160.0351668 - 164.2955264 i, yo = 22.65249665 - 23.88749381 i, R = 0 \},
\{xo = 160.0351668 + 164.2955264 i, yo = 22.65249665 + 23.88749381 i, R = 0 \},
\{yo = 23.29397054 - 96.54309533 i, xo = 162.8737597 + 13.89952224 i, R = 0 \},
\{xo = 162.8737597 - 13.89952224 i, yo = 23.29397054 + 96.54309533 i, R = 0 \},
\{xo = 160.2866220, yo = 35.87926258, R = 0 \},
\{yo = 420.5485839, xo = 105.2358671, R = -408.6224474 \},
\{yo = 420.5485839, xo = 105.2358671, R = 408.6224474 \}

> circ := (x-xo)^2 + (y-yo)^2 = R^2;

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
fieldplot, fieldplot3d, gridplot, gridplot3d, implicitplot, implicitplot3d, inequal, listcontourplot,
listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
poinplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
polyhedraplot, replplot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematrixplot, sphereplot, surfdata, texplot, texplot3d, tubeplot]

> p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);

> with(stats[statplots]);

[boxplot, histogram, scatterplot, xscale, xshift, yexchange, zexchange, yscale, yshift,
 yexchange, zscale, zshift]

> p2 := scatterplot(ww, yy, color=black);

> display([p1, p2], scaling=constrained);
$R = 4.08 \text{ m}$
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \(xx\), and the corresponding y coordinates are given as a list \(yy\). The number of data points is \(n\). The data I entered was labelled. 

\[
\text{restart;}
\]

\[
xx := [10, 40, 70, 100, 130, 160, 190, 220, 250, 280, 310];
\]
\[
xx := [10, 40, 70, 100, 130, 160, 190, 220, 250, 280, 310];
\]

\[
yy := [22.1, 16.1, 11.3, 14.1, 16.6, 20.2, 20.4, 38.9, 51.2, 65.6];
\]
\[
yy := [22.1, 16.1, 11.3, 14.1, 16.6, 20.2, 20.4, 38.9, 51.2, 65.6];
\]

\[
n := 11;
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0,\) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]

\[
eq 3 := \text{diff}(F(x_0, y_0, R), R) = 0;
\]

\[
eq 4 := \text{diff}(F(x_0, y_0, R), R) = 0;
\]
\[ + 2((130 - x_0)^2 + (14 - y_0)^2 - R^2)(-28 + 2y_0) \\
+ 2((160 - x_0)^2 + (16.8 - y_0)^2 - R^2)(-33.2 + 2y_0) \\
+ 2((190 - x_0)^2 + (20 - y_0)^2 - R^2)(-40 + 2y_0) \\
+ 2((220 - x_0)^2 + (28.4 - y_0)^2 - R^2)(-56.8 + 2y_0) \\
+ 2((250 - x_0)^2 + (38.9 - y_0)^2 - R^2)(-77.8 + 2y_0) \\
+ 2((280 - x_0)^2 + (51.2 - y_0)^2 - R^2)(-102.4 + 2y_0) \\
+ 2((310 - x_0)^2 + (65.6 - y_0)^2 - R^2)(-131.2 + 2y_0) = 0 \]

\texttt{eq3 := \texttt{diff(F(x0,y0,R),R)} = 0;} \\
\texttt{eq3 := -4(10 - x_0)^2 + (22.1 - y_0)^2 - R^2) R - 4((40 - x_0)^2 + (16 - y_0)^2 - R^2) R} \\
- 4((70 - x_0)^2 + (11 - y_0)^2 - R^2) R - 4((100 - x_0)^2 + (13 - y_0)^2 - R^2) R} \\
- 4((130 - x_0)^2 + (14 - y_0)^2 - R^2) R - 4((160 - x_0)^2 + (16.6 - y_0)^2 - R^2) R \\
- 4((190 - x_0)^2 + (20 - y_0)^2 - R^2) R - 4((220 - x_0)^2 + (28.4 - y_0)^2 - R^2) R} \\
- 4((250 - x_0)^2 + (38.9 - y_0)^2 - R^2) R - 4((280 - x_0)^2 + (51.2 - y_0)^2 - R^2) R} \\
- 4((310 - x_0)^2 + (65.6 - y_0)^2 - R^2) R = 0 \]

\texttt{sol := \texttt{solve}\{(eq1, eq2, eq3), (x0, y0, R)\};} \\
\texttt{sol := \{x0 = 160.0127321 - 164.6313913 I, y0 = 22.59550876 - 24.18934278 I, R = 0\},} \\
\texttt{(y0 = 22.59550876 + 24.18934278 I, R = 0, x0 = 160.0127321 + 164.6313913 I \},} \\
\texttt{\{x0 = 160.0336610, y0 = 35.35361892. R = 0\},} \\
\texttt{(y0 = 23.16165218 + 96.59658323 I, x0 = 162.4827054 - 14.07799857 I, R = 0 \},} \\
\texttt{(y0 = 23.16165218 - 96.59658323 I, x0 = 162.4827054 + 14.07799857 I, R = 0 \},} \\
\texttt{(x0 = 101.0230725, y0 = 442.1887862, R = -430.3031155 \},} \\
\texttt{(R = 430.3031155, x0 = 101.0230725, y0 = 442.1887862 \}} \\
\texttt{\texttt{circ := (x - x_0)^2 + (y - y_0)^2 - R^2;}} \\
\texttt{\texttt{circ := (x - x_0)^2 + (y - y_0)^2 = R^2}} \\
\texttt{\texttt{with(plots);}} \\
\texttt{[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,} \\
\texttt{conourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,} \\
\texttt{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontourplot,} \\
\texttt{listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,} \\
\texttt{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,} \\
\texttt{polyhedrplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,} \\
\texttt{sparsematrixplot, sphereplot, surfdata, texture, textplot, textplot3d, tubeplot]}} \\
\texttt{\texttt{pi1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);}} \\
\texttt{\texttt{with(stats[statplots]);}}
> p2 := scatterplot(xx, yy, color=black):
> display([p1, p2], scaling=constrained);

\[ R = 4.3 \text{ m} \]
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, «enter data».

\[
\text{restart};
\]

\[
xx := \{50, 100, 150, 200, 250\};
\]

\[
xx := \{50, 100, 150, 200, 250\}
\]

\[
yy := \{13.7, 13, 16, 22.8, 38.9\};
\]

\[
yy := \{13.7, 13, 16, 22.8, 38.9\}
\]

\[
n := 5;
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\), and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\), and \(R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 1 := 2 ((50 - x_0)^2 + (13.7 - y_0)^2 - R^2) (-100 + 2 x_0)
\]

\[
+ 2 ((100 - x_0)^2 + (13 - y_0)^2 - R^2) (-200 + 2 x_0)
\]

\[
+ 2 ((150 - x_0)^2 + (16 - y_0)^2 - R^2) (-300 + 2 x_0)
\]

\[
+ 2 ((200 - x_0)^2 + (22.8 - y_0)^2 - R^2) (-400 + 2 x_0)
\]

\[
+ 2 ((250 - x_0)^2 + (38.9 - y_0)^2 - R^2) (-500 + 2 x_0) = 0
\]

\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]

\[
eq 2 := 2 ((50 - x_0)^2 + (13.7 - y_0)^2 - R^2) (-27.4 + 2 y_0)
\]

\[
+ 2 ((100 - x_0)^2 + (13 - y_0)^2 - R^2) (-26 + 2 y_0)
\]

\[
+ 2 ((150 - x_0)^2 + (16 - y_0)^2 - R^2) (-32 + 2 y_0)
\]

\[
+ 2 ((200 - x_0)^2 + (22.8 - y_0)^2 - R^2) (-45.6 + 2 y_0)
\]

\[
+ 2 ((250 - x_0)^2 + (38.9 - y_0)^2 - R^2) (-77.8 + 2 y_0) = 0
\]

\[
eq 3 := \text{diff}(F(x_0, y_0, R), R) = 0;
\]

\[
eq 3 := -4 ((50 - x_0)^2 + (13.7 - y_0)^2 - R^2) R
\]

\[
- 4 ((100 - x_0)^2 + (13 - y_0)^2 - R^2) R
\]

\[
- 4 ((150 - x_0)^2 + (16 - y_0)^2 - R^2) R
\]

\[
- 4 ((200 - x_0)^2 + (22.8 - y_0)^2 - R^2) R
\]

\[
- 4 ((250 - x_0)^2 + (38.9 - y_0)^2 - R^2) R = 0
\]

\[
sol := \text{solve}\{\{\text{eq1}, \text{eq2}, \text{eq3}\}, \{x_0, y_0, R\}\};
\]
\textit{sol := \{R = 0, x0 = 149.9905602 - 122.5664414 I, y0 = 18.99006090 - 14.82835188 I\}},
\{R = 0, x0 = 149.9905602 + 122.5664414 I, y0 = 18.99006090 + 14.82835188 I\},
\{R = 0, x0 = 150.0272157, y0 = 24.59107457 \},
\{R = 0, x0 = 150.9288283 - 8.590008460 I, y0 = 19.13831291 + 71.20988802 I\},
\{R = 0, x0 = 150.9288283 + 8.590008460 I, y0 = 19.13831291 - 71.20988802 I\},
\{x0 = 94.95342302, y0 = 482.1300414, R = -469.9732970 \},
\{x0 = 94.95342302, R = 469.9732970, y0 = 482.1300414 \}

\textit{circ := (x-x0)^2 + (y-y0)^2 = R^2 ;}

\textit{with(plots);}

\textit{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformat,}
\textit{contourplot, contourplot3d, coordplot, coodplot3d, cylinderplot, densityplot, display, display3d,}
\textit{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, liscontplot,}
\textit{liscontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,}
\textit{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedral_supported,}
\textit{polyhedralplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,}
\textit{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot}

\textit{p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);}

\textit{with(statplots[statplot]);}

\textit{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,}
\textit{yexchange, xscale, xshift}

\textit{p2 := scatterplot(xx, yy, color=black);}

\textit{display([p1, p2], scaling=constrained);}
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was given as:

\[
\begin{align*}
xx &: = [50, 150, 250] \\
yy &: = [13.7, 16, 38.9] \\
n &: = 3
\end{align*}
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\):

\[
F = (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((xx[i] - x_0)^2 + (yy[i] - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, \text{ and } R\).

```plaintext
> eq1 := diff(F(x0, y0, R), x0) = 0;
> eq2 := diff(F(x0, y0, R), y0) = 0;
> eq3 := diff(F(x0, y0, R), R) = 0;
> sol := solve({eq1, eq2, eq3}, {x0, y0, R});
```

\[
\begin{align*}
sol &: = \{ R = 0, x0 = 150.0010613 - 141.5078106 i, y0 = 21.15055718 - 17.90003749 i \}, \\
{ R = 0, x0 = 21.15055718 + 17.90003749 i, y0 = 150.0010613 + 141.5078106 i } , \\
{ R = 0, x0 = 26.24758805, y0 = 150.00043718 } , \\
{ R = 0, x0 = 150.8512194 - 10.36995806 i, y0 = 21.28406377 + 82.12070756 i } , \\
{ R = 0, y0 = 21.28406377 - 82.12070756 i, x0 = 150.8512194 + 10.36995806 i } , \\
x0 &= 88.51279515, y0 = 514.2936893, R = -502.0729799
\end{align*}
\]
\[
\{ x_0 = 88.51279515, y_0 = 514.2936893, R = 502.0729799 \}
\]

\[
\text{circ} := (x - x_0)^2 + (y - y_0)^2 = R^2
\]

\[\text{with}(\text{plots});\]

\{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontourplot, listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra-supported, polyhedrplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot\}

\[\text{p1} := \text{implicitplot}(\text{subs(sol[7], circ)}, x=0..400, y=-300..300, \text{numpoints}=4000);\]

\[\text{with}(\text{stats[statplots]});\]

\{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift, yxexchange, zscale, zshift\}

\[\text{p2} := \text{scatterplot}(xx, yy, \text{color=black});\]

\[\text{display}([\text{p1}, \text{p2}], \text{scaling=constrained});\]
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as \( xx \), and the list of y coordinates as \( yy \). The number of points in the list is \( n \). The index of the loop is \( i \), which refers to the index of the first of the three points in the list. The output of the loop at each step is \( i \), and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative \( R \). If the three points are co-linear, then no solution can be found, so the loop simply outputs the value of \( i \). The data I entered was labelled, "Futrech VIII B3E".

```plaintext
> restart;

\( xx := \{ 7.2, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 316 \} ;

\( yy := \{ 28.3, 22.1, 20.7, 17.6, 16.1, 13.7, 11.9, 11.1, 12.4, 12.8, 13.3, 13.5, 14.3, 14.5, 15.6, 16.6, 17.4, 20.8, 25.3, 28.4, 32.0, 35.5, 38.9, 42.8, 47.0, 51.2, 55.4, 60.2, 65.6, 71.1 \} ;

\( cir := (x, y) \rightarrow ((x-x0)^2 + (y-y0)^2 = R^2 ;

\( n := 33 ;

\( cir := (x, y) \rightarrow (x - x0)^2 + (y - y0)^2 = R^2

> for i from 1 to n-2 do
  > eq1 := cir(xx[i], yy[i]) ;
  > eq2 := cir(xx[i+1], yy[i+1]) ;
  > eq3 := cir(xx[i+2], yy[i+2]) ;
  > print(i, solve({eq1, eq2, eq3}, {x0, y0, R})) ;
  > od;
```

```
1, \{ y0 = 28.54187328, x0 = 15.99986226, R = -8.803185699 \}.
   \{ y0 = 28.54187328, x0 = 15.99986226, R = 8.803185699 \}
2, \{ y0 = -41.52647059, x0 = 6.190294118, R = -63.74042374 \},
   \{ y0 = -41.52647059, x0 = 6.190294118, R = 63.74042374 \}
3, \{ x0 = 46.44373333, y0 = 88.32333333, R = -72.60982195 \},
   \{ x0 = 46.44373333, y0 = 88.32333333, R = 72.60982195 \}
4, \{ y0 = -132.4642857, R = -151.2475803, x0 = 11.1771429 \},
   \{ R = 151.2475803, y0 = -132.4642857, x0 = 11.1771429 \}
5, \{ y0 = 222.230000, x0 = 92.69740000, R = -212.8563574 \},
   \{ y0 = 222.230000, x0 = 92.69740000, R = 212.8563574 \}
6, \{ y0 = 125.2611111, x0 = 75.24300000, R = -114.3813383 \},
   \{ y0 = 125.2611111, x0 = 75.24300000, R = 114.3813383 \}
7, \{ y0 = 55.08042478, x0 = 68.92673913, R = -13 \}.
```

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\[ \{ y_0 = 55.08043478, x_0 = 68.92673913, R = 44.09349861 \} \]
\[ \{ x_0 = 89.05040000, y_0 = -88.66000000, R = -101.4644437 \} \]
\[ \{ x_0 = 89.05040000, y_0 = -88.66000000, R = 101.4644437 \} \]
\[ \{ x_0 = 105.0120000, y_0 = -487.7000000, R = -500.7250844 \} \]
\[ \{ x_0 = 105.0120000, y_0 = -487.7000000, R = 500.7250844 \} \]
\[ \{ y_0 = 346.8166667, x_0 = 88.32166667, R = -334.0208832 \} \]
\[ \{ y_0 = 346.8166667, x_0 = 88.32166667, R = 334.0208832 \} \]
\[ \{ R = -335.5225891, x_0 = 88.24666667, y_0 = 348.3166667 \} \]
\[ \{ R = 335.5225891, x_0 = 88.24666667, y_0 = 348.3166667 \} \]
\[ \{ x_0 = 122.2672727, y_0 = -76.94090909, R = -91.26907481 \} \]
\[ \{ x_0 = 122.2672727, y_0 = -76.94090909, R = 91.26907481 \} \]
\[ \{ R = -77.38887293, y_0 = 91.34230769, x_0 = 127.3157692 \} \]
\[ \{ y_0 = 91.34230769, x_0 = 127.3157692, R = 77.38887293 \} \]
\[ 14 \]
\[ \{ y_0 = -235.7000000, x_0 = 170.1200000, R = -252.5028800 \} \]
\[ \{ y_0 = -235.7000000, x_0 = 170.1200000, R = 252.5028800 \} \]
\[ \{ x_0 = 158.7575000, y_0 = -46.3250000, R = -62.93726584 \} \]
\[ \{ R = 62.93726584, x_0 = 158.7575000, y_0 = -46.3250000 \} \]
\[ \{ R = -36.49839873, x_0 = 168.5971429, y_0 = 52.07142857 \} \]
\[ \{ R = 36.49839873, x_0 = 168.5971429, y_0 = 52.07142857 \} \]
\[ \{ x_0 = 151.2130000, y_0 = 148.6500000, R = -134.3698399 \} \]
\[ \{ x_0 = 151.2130000, y_0 = 148.6500000, R = 134.3698399 \} \]
\[ \{ x_0 = 45.17200000, y_0 = 556.5000000, R = -555.7044175 \} \]
\[ \{ x_0 = 45.17200000, y_0 = 556.5000000, R = 555.7044175 \} \]
\[ \{ x_0 = 294.5166667, y_0 = -334.0166667, R = -369.1226543 \} \]
\[ \{ x_0 = 294.5166667, y_0 = -334.0166667, R = 369.1226543 \} \]
\[ \{ y_0 = 205.1833333, R = -186.7790867, x_0 = 159.7166667 \} \]
\[ \{ y_0 = 205.1833333, x_0 = 159.7166667, R = 186.7790867 \} \]
\[ \{ y_0 = 250.9700000, x_0 = 145.5228000, R = -234.7003584 \} \]
\[ \{ y_0 = 250.9700000, x_0 = 145.5228000, R = 234.7003584 \} \]
\[ \{ y_0 = -1094.0500000, R = -1194.894420, x_0 = 629.7300000 \} \]
\[ \{ y_0 = -1094.0500000, R = 1194.894420, x_0 = 629.7300000 \} \]
\[ \{ x_0 = 626.0550000, y_0 = -1083.550000, R = -1183.769980 \} \]
\[ \{ x_0 = 626.0550000, y_0 = -1083.550000, R = 1183.769980 \} \]
\[ \{ x_0 = 167.3202000, y_0 = 265.6700000, R = -241.3722897 \} \]
\[ \{ x_0 = 167.3202000, y_0 = 265.6700000, R = 241.3722897 \} \]
26. \( \{ R = 418.6802699, x0 = 102.8870000, y0 = 430.8833333 \} \).
\[ R = 418.6802699, x0 = 102.8870000, y0 = 430.8833333 \]

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28

29. \( \{ x0 = 199.8800000, y0 = 255.9666667, R = -219.8831557 \} \).
\( \{ x0 = 199.8800000, y0 = 255.9666667, R = 219.8831557 \} \)

30. \( \{ x0 = 192.9680000, y0 = 270.3666667, R = -235.8513871 \} \).
\( \{ x0 = 192.9680000, y0 = 270.3666667, R = 235.8513871 \} \)

31. \( \{ y0 = 98.62222222, R = -40.99355145, x0 = 285.7100000 \} \).
\( \{ y0 = 98.62222222, x0 = 285.7100000, R = 40.99355145 \} \)
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx, and the list of y coordinates as yy. The number of points in the list is n. The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative R. If the three points are colinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "Putlock VIII, B3E".

```matlab
> restart;
> xx := [10, 40, 70, 100, 130, 160, 190, 220, 250, 280, 310];
> xx := [10, 40, 70, 100, 130, 160, 190, 220, 250, 280, 310]
> yy := [22.1, 16, 11, 13, 14, 16.6, 20, 28.4, 38.9, 51.2, 65.6];
> yy := [22.1, 16, 11, 13, 14, 16.6, 20, 28.4, 38.9, 51.2, 65.6]
> cir := (x, y) -> ((x-x0)^2+(y-y0)^2)=R^2;
> cir := (x, y) -> (x-x0)^2 + (y-y0)^2 = R^2
> n := 11;
> n := 11
> for i from 1 to n-2 do
> eq1 := cir(xx[i], yy[i]);
> eq2 := cir(xx[i+1], yy[i+1]);
> eq3 := cir(xx[i+2], yy[i+2]);
> print(i, solve({eq1, eq2, eq3}, (x0, y0, R)));
> od:
```

1. \{ y0 = 862.4590909, R = 860.8037573, x0 = 196.4931818 \},
   \{ y0 = 862.4590909, x0 = 196.4931818, R = 860.8037573 \}

2. \{ y0 = \frac{1983}{14}, x0 = \frac{1069}{14}, R = \frac{1}{7} \text{ RootOf} (2 \_ Z^2 - 1676581) \}

3. \{ y0 = \frac{-1779}{2}, x0 = \frac{1451}{10}, R = \frac{1}{5} \text{ RootOf} (2 \_ Z^2 - 40827013) \}

4. \{ y0 = 578.9250000, x0 = 96.15250000, R = -565.9380787 \},
   \{ y0 = 578.9250000, x0 = 96.15250000, R = 565.9380787 \}

5. \{ y0 = 1153.050000, x0 = 46.39500000, R = -1142.114136 \},
   \{ y0 = 1153.050000, x0 = 46.39500000, R = 1142.114136 \}

6. \{ y0 = 208.2120000, x0 = 153.4766400, R = -191.7230105 \},
   \{ y0 = 208.2120000, x0 = 153.4766400, R = 191.7230105 \}

7. \{ y0 = 500.0214286, x0 = 71.77000000, R = -494.3671761 \},
   \{ R = 494.3671761, y0 = 500.0214286, x0 = 71.77000000 \}

8. \{ R = -612.4802876, y0 = 611.5500000, x0 = 32.73500000 \},
   \{ y0 = 611.5500000, x0 = 32.73500000, R = 612.4802876 \}

9. \{ R = -562.3663528, y0 = 565.1642857, x0 = 51.75314286 \}

Page 1
( R = 562.3663528, y0 = 565.1642857, \phi = 51.75314286 )
C4
Port side
Floor VIII
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list $xx$, and the corresponding y coordinates are given as a list $yy$. The number of data points is $n$. The data I entered was labelled, "Putlock VIII, BE".

```maple
restart;
xx := [30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160];
xx := [30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160];

yy := [.2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2];

yy := [.2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2, .2];

n := 14;
```

If we find a circle $(x - x_0)^2 + (y - y_0)^2 = R^2$ which passes exactly though the given points $[x_i, y_i]$ for $i = 1 .. n$, then the $n$ quantities $(x_i - x_0)^2 + (y_i - y_0)^2 - R^2$ will all be zero. This is too much to hope for, but we can seek three numbers $x_0, y_0$, and $R$ so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function $F$.

```maple
F := (x0, y0, r) -> sum((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2, i = 1 .. n;
```

We seek a critical point for the function $F$ by setting its partial derivatives equal to zero, and solving the resulting nonlinear system of three equations in the three unknowns $x_0, y_0$, and $R$.

```maple
eq 1 := diff(F(x0, y0, R), x0) = 0;
eq 2 := diff(F(x0, y0, R), y0) = 0;
```

```maple
eq 1 := 2 ((30 - x0)^2 + (.2 - y0)^2 - R^2) (-60 + 2 x0) + 2 ((40 - x0)^2 + (.2 - y0)^2 - R^2) (-80 + 2 x0) - 2 ((50 - x0)^2 + (.3 - y0)^2 - R^2) (-100 + 2 x0) + 2 ((60 - x0)^2 + (.4 - y0)^2 - R^2) (-120 + 2 x0) + 2 ((70 - x0)^2 + (.5 - y0)^2 - R^2) (-140 + 2 x0) + 2 ((80 - x0)^2 + (.6 - y0)^2 - R^2) (-160 + 2 x0) + 2 ((90 - x0)^2 + (.7 - y0)^2 - R^2) (-180 + 2 x0) + 2 ((100 - x0)^2 + (.8 - y0)^2 - R^2) (-200 + 2 x0) + 2 ((110 - x0)^2 + (.9 - y0)^2 - R^2) (-220 + 2 x0) + 2 ((120 - x0)^2 + (10 - y0)^2 - R^2) (-240 + 2 x0) + 2 ((130 - x0)^2 + (11 - y0)^2 - R^2) (-260 + 2 x0) + 2 ((140 - x0)^2 + (12 - y0)^2 - R^2) (-280 + 2 x0) + 2 ((150 - x0)^2 + (13 - y0)^2 - R^2) (-300 + 2 x0) + 2 ((160 - x0)^2 + (14 - y0)^2 - R^2) (-320 + 2 x0) = 0;
eq 2 := 2 ((30 - x0)^2 + (.2 - y0)^2 - R^2) (-4 + 2 y0) + 2 ((40 - x0)^2 + (.2 - y0)^2 - R^2) (-4 + 2 y0) + 2 ((50 - x0)^2 + (.3 - y0)^2 - R^2) (-6 + 2 y0) + 2 ((60 - x0)^2 + (.4 - y0)^2 - R^2) (-8 + 2 y0) + 2 ((70 - x0)^2 + (.5 - y0)^2 - R^2) (-10 + 2 y0) + 2 ((80 - x0)^2 + (.6 - y0)^2 - R^2) (-12 + 2 y0) + 2 ((90 - x0)^2 + (.7 - y0)^2 - R^2) (-14 + 2 y0) + 2 ((100 - x0)^2 + (.8 - y0)^2 - R^2) (-16 + 2 y0) + 2 ((110 - x0)^2 + (.9 - y0)^2 - R^2) (-18 + 2 y0) + 2 ((120 - x0)^2 + (10 - y0)^2 - R^2) (-20 + 2 y0) + 2 ((130 - x0)^2 + (11 - y0)^2 - R^2) (-22 + 2 y0) + 2 ((140 - x0)^2 + (12 - y0)^2 - R^2) (-24 + 2 y0) + 2 ((150 - x0)^2 + (13 - y0)^2 - R^2) (-26 + 2 y0) + 2 ((160 - x0)^2 + (14 - y0)^2 - R^2) (-28 + 2 y0) = 0;
eq 3 := 2 ((30 - x0)^2 + (.2 - y0)^2 - R^2) (-8 + 2 y0) + 2 ((40 - x0)^2 + (.2 - y0)^2 - R^2) (-10 + 2 y0) + 2 ((50 - x0)^2 + (.3 - y0)^2 - R^2) (-12 + 2 y0) + 2 ((60 - x0)^2 + (.4 - y0)^2 - R^2) (-14 + 2 y0) + 2 ((70 - x0)^2 + (.5 - y0)^2 - R^2) (-16 + 2 y0) + 2 ((80 - x0)^2 + (.6 - y0)^2 - R^2) (-18 + 2 y0) + 2 ((90 - x0)^2 + (.7 - y0)^2 - R^2) (-20 + 2 y0) + 2 ((100 - x0)^2 + (.8 - y0)^2 - R^2) (-22 + 2 y0) + 2 ((110 - x0)^2 + (.9 - y0)^2 - R^2) (-24 + 2 y0) + 2 ((120 - x0)^2 + (10 - y0)^2 - R^2) (-26 + 2 y0) + 2 ((130 - x0)^2 + (11 - y0)^2 - R^2) (-28 + 2 y0) + 2 ((140 - x0)^2 + (12 - y0)^2 - R^2) (-30 + 2 y0) + 2 ((150 - x0)^2 + (13 - y0)^2 - R^2) (-32 + 2 y0) + 2 ((160 - x0)^2 + (14 - y0)^2 - R^2) (-34 + 2 y0) = 0;
```
\[
+2 ((60 - x0)^2 + (x - y0)^2 - R^2) (-8 + 2 y0) + 2 ((70 - x0)^2 + (y - y0)^2 - R^2) (-4 + 2 y0)
+4 ((80 - x0)^2 - y0^2 - R^2) y0 + 2 ((90 - x0)^2 + (1 - y0)^2 - R^2) (-2 + 2 y0)
+2 ((100 - x0)^2 + (0.8 - y0)^2 - R^2) (-1.6 + 2 y0)
+2 ((110 - x0)^2 + (2 - y0)^2 - R^2) (-4 + 2 y0)
+2 ((120 - x0)^2 + (3.3 - y0)^2 - R^2) (-6.6 + 2 y0)
+2 ((130 - x0)^2 + (5.1 - y0)^2 - R^2) (-10.2 + 2 y0)
+2 ((140 - x0)^2 + (7.5 - y0)^2 - R^2) (-15.0 + 2 y0)
+2 ((150 - x0)^2 + (10 - y0)^2 - R^2) (-20 + 2 y0)
+2 ((160 - x0)^2 + (13.4 - y0)^2 - R^2) (-26.8 + 2 y0) = 0
\]

> \texttt{eq3 := diff(F(x0, y0, R, R), R) = 0;} \\
\texttt{eq3 := -4 ((30 - x0)^2 + (2 - y0)^2 - R^2) R - 4 ((40 - x0)^2 + (2 - y0)^2 - R^2) R}
-4 ((50 - x0)^2 + (3 - y0)^2 - R^2) R - 4 ((60 - x0)^2 + (4 - y0)^2 - R^2) R
-4 ((70 - x0)^2 + (2 - y0)^2 - R^2) R - 4 ((80 - x0)^2 + y0^2 - R^2) R
-4 ((90 - x0)^2 + (1.1 - y0)^2 - R^2) R - 4 ((100 - x0)^2 + (8 - y0)^2 - R^2) R
-4 ((110 - x0)^2 + (2 - y0)^2 - R^2) R - 4 ((120 - x0)^2 + (3.3 - y0)^2 - R^2) R
-4 ((130 - x0)^2 + (5.1 - y0)^2 - R^2) R - 4 ((140 - x0)^2 + (7.5 - y0)^2 - R^2) R
-4 ((150 - x0)^2 + (10 - y0)^2 - R^2) R - 4 ((160 - x0)^2 + (13.4 - y0)^2 - R^2) R = 0

> \texttt{sol := solve([eq1,eq2,eq3], (x0,y0,R));}

\texttt{sol := [R = 0, y0 = 2.157425966 - 6.152564211 i, x0 = 94.99351482 + 69.86053328 i],}
\texttt{[R = 0, y0 = 2.157425966 + 6.152564211 i, x0 = 94.99351482 + 69.86053328 i],}
\texttt{[R = 0, x0 = 95.01523556, y0 = 4.980886152].}
\texttt{[R = 0, x0 = 95.35116283 - 3.560146176 i, y0 = 2.201831153 + 40.52015788 i],}
\texttt{[R = 0, x0 = 95.35116283 + 3.560146176 i, y0 = 2.201831153 - 40.52015788 i],}
\texttt{(y0 = 325.2265709, x0 = 66.91875280, R = -325.8708467),}
\texttt{[y0 = 325.2265709, x0 = 66.91875280, R = 325.8708467,]
> \texttt{circ := (x-x0)^2 + (y-y0)^2 = R^2}

> \texttt{with(plots);}

\texttt{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,}
\texttt{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,}
\texttt{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,}
\texttt{listcontplot3d, listdensityplot, listplot, limplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,}
\texttt{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedron, supported,}
\texttt{polyhedroplot, repplot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,}
\texttt{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]}

Page 2
\[ R = 3.25 \text{ m} \]
B4E
Port side
Futtock VIII
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list `xx`, and the corresponding y coordinates are given as a list `yy`. The number of data points is `n`. The data I entered was labelled, 

```maple
> restart:
> xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 311];
> yy := [105.5, 97.2, 89.2, 83, 75.6, 68.8, 63.1, 57.2, 51.9, 46.7, 41.3, 35.5, 30.1, 26.1, 22.6, 18.7, 14.9, 11.8, 9.62, 4.1, 2.8, 1.5, 6.3, 0.2, 2, 1.5, 4.67, 1.89, 10];
> n := 33;
```

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

```maple
> F := (x0, y0, R) -> sum(((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2, i = 1 .. n);
```

\[
F := (x0, y0, R) \rightarrow \sum_{i=1}^{n} ((xx_i - x0)^2 + (yy_i - y0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\).

```maple
> eq1 := diff(F(x0, y0, R), x0) = 0;
> eq2 := diff(F(x0, y0, R), y0) = 0;
> eq3 := diff(F(x0, y0, R), R) = 0;
```

```maple
eq 1 := 2 ((280 - x0)^2 + (1.5 - y0)^2 - R^2) (-560 + 2 x0) + 2 ((290 - x0)^2 + (4.6 - y0)^2 - R^2) (-580 + 2 x0) + 2 ((300 - x0)^2 + (7 - y0)^2 - R^2) (-600 + 2 x0) + 2 ((310 - x0)^2 + (9.8 - y0)^2 - R^2) (-620 + 2 x0) + 2 ((311 - x0)^2 + (10 - y0)^2 - R^2) (-622 + 2 x0) + 4 (x0^2 + (105.5 - y0)^2 - R^2) x0 + 2 ((10 - x0)^2 + (97.2 - y0)^2 - R^2) (-20 + 2 x0) + 2 ((20 - x0)^2 + (89.2 - y0)^2 - R^2) (-40 + 2 x0) + 2 ((30 - x0)^2 + (83 - y0)^2 - R^2) (-60 + 2 x0) + 2 ((40 - x0)^2 + (75.6 - y0)^2 - R^2) (-80 + 2 x0) + 2 ((50 - x0)^2 + (68.8 - y0)^2 - R^2) (-100 + 2 x0) + 2 ((60 - x0)^2 + (63.1 - y0)^2 - R^2) (-120 + 2 x0)
}\]
+ 2 ((70 - x0)^2 + (57.2 - y0)^2 - R^2) (-140 + 2 x0) \\
- 2 ((80 - x0)^2 + (51.9 - y0)^2 - R^2) (-160 + 2 x0) \\
+ 2 ((90 - x0)^2 + (46.7 - y0)^2 - R^2) (-180 + 2 x0) \\
+ 2 ((100 - x0)^2 + (41.3 - y0)^2 - R^2) (-200 + 2 x0) \\
+ 2 ((110 - x0)^2 + (35.5 - y0)^2 - R^2) (-220 + 2 x0) \\
+ 2 ((260 - x0)^2 + (0.2 - y0)^2 - R^2) (-520 + 2 x0) \\
+ 2 ((270 - x0)^2 + (0.2 - y0)^2 - R^2) (-540 + 2 x0) + 2 ((250 - x0)^2 + y0^2 - R^2) (-500 + 2 x0) \\
+ 2 ((210 - x0)^2 + (2.8 - y0)^2 - R^2) (-420 + 2 x0) \\
+ 2 ((220 - x0)^2 + (1.5 - y0)^2 - R^2) (-440 + 2 x0) \\
+ 2 ((230 - x0)^2 + (0.6 - y0)^2 - R^2) (-460 + 2 x0) \\
+ 2 ((240 - x0)^2 + (0.3 - y0)^2 - R^2) (-480 + 2 x0) \\
+ 2 ((120 - x0)^2 + (30.1 - y0)^2 - R^2) (-240 + 2 x0) \\
+ 2 ((130 - x0)^2 + (26.1 - y0)^2 - R^2) (-260 + 2 x0) \\
+ 2 ((140 - x0)^2 + (22.6 - y0)^2 - R^2) (-280 + 2 x0) \\
+ 2 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) (-300 + 2 x0) \\
+ 2 ((160 - x0)^2 + (14.9 - y0)^2 - R^2) (-320 + 2 x0) \\
+ 2 ((170 - x0)^2 + (11.8 - y0)^2 - R^2) (-340 + 2 x0) \\
+ 2 ((180 - x0)^2 + (9 - y0)^2 - R^2) (-360 + 2 x0) \\
+ 2 ((190 - x0)^2 + (6.2 - y0)^2 - R^2) (-380 + 2 x0) \\
+ 2 ((200 - x0)^2 + (4.1 - y0)^2 - R^2) (-400 + 2 x0) = 0 \\
> eq2 := \text{diff}(E(x0,y0,R),y0) = 0; \\
eq q2 := 2 ((280 - x0)^2 + (1.5 - y0)^2 - R^2) (-3.0 + 2 y0) \\
+ 2 ((290 - x0)^2 + (4.6 - y0)^2 - R^2) (-9.2 + 2 y0) \\
+ 2 ((300 - x0)^2 + (7 - y0)^2 - R^2) (-14 + 2 y0) \\
+ 2 ((310 - x0)^2 + (9.8 - y0)^2 - R^2) (-19.6 + 2 y0) \\
+ 2 ((320 - x0)^2 + (10 - y0)^2 - R^2) (-20 + 2 y0) \\
+ 2 (x0^2 + (105.5 - y0)^2 - R^2) (-211.0 + 2 y0) \\
+ 2 ((10 - x0)^2 + (97.2 - y0)^2 - R^2) (-194.4 + 2 y0) \\
+ 2 ((20 - x0)^2 + (89.2 - y0)^2 - R^2) (-178.4 + 2 y0) \\
+ 2 ((30 - x0)^2 + (83 - y0)^2 - R^2) (-166 + 2 y0) \\
+ 2 ((40 - x0)^2 + (75.6 - y0)^2 - R^2) (-151.2 + 2 y0) \\
+ 2 ((50 - x0)^2 + (68.8 - y0)^2 - R^2) (-137.6 + 2 y0) \\
+ 2 ((60 - x0)^2 + (63.1 - y0)^2 - R^2) (-126.2 + 2 y0) \\
>
\[\begin{align*}
+ 2 \left( (70 - x_0)^2 + (57.2 - y_0)^2 - R^2 \right) (-114.4 + 2 y_0) \\
+ 2 \left( (80 - x_0)^2 + (51.9 - y_0)^2 - R^2 \right) (-103.8 + 2 y_0) \\
+ 2 \left( (90 - x_0)^2 + (46.7 - y_0)^2 - R^2 \right) (-93.4 + 2 y_0) \\
+ 2 \left( (100 - x_0)^2 + (41.3 - y_0)^2 - R^2 \right) (-82.6 + 2 y_0) \\
+ 2 \left( (110 - x_0)^2 + (35.5 - y_0)^2 - R^2 \right) (-71.0 + 2 y_0) \\
+ 2 \left( (120 - x_0)^2 + (30.1 - y_0)^2 - R^2 \right) (-60.2 + 2 y_0) \\
+ 2 \left( (130 - x_0)^2 + (26.1 - y_0)^2 - R^2 \right) (-52.2 + 2 y_0) \\
+ 2 \left( (140 - x_0)^2 + (22.6 - y_0)^2 - R^2 \right) (-45.2 + 2 y_0) \\
+ 2 \left( (150 - x_0)^2 + (18.7 - y_0)^2 - R^2 \right) (-37.4 + 2 y_0) \\
+ 2 \left( (160 - x_0)^2 + (14.9 - y_0)^2 - R^2 \right) (-29.8 + 2 y_0) \\
+ 2 \left( (170 - x_0)^2 + (11.8 - y_0)^2 - R^2 \right) (-23.6 + 2 y_0) \\
+ 2 \left( (180 - x_0)^2 + (9.0 - y_0)^2 - R^2 \right) (-18 + 2 y_0) \\
+ 2 \left( (270 - x_0)^2 + (2.0 - y_0)^2 - R^2 \right) (-4 + 2 y_0) \\
+ 2 \left( (280 - x_0)^2 + (3.0 - y_0)^2 - R^2 \right) (-4 + 2 y_0) + 4 \left( (250 - x_0)^2 + (3.0 - y_0)^2 - R^2 \right) y_0 \\
+ 2 \left( (190 - x_0)^2 + (6.2 - y_0)^2 - R^2 \right) (-12.4 + 2 y_0) \\
+ 2 \left( (200 - x_0)^2 + (4.1 - y_0)^2 - R^2 \right) (-8.2 + 2 y_0) \\
+ 2 \left( (210 - x_0)^2 + (2.8 - y_0)^2 - R^2 \right) (-5.6 + 2 y_0) \\
+ 2 \left( (220 - x_0)^2 + (1.5 - y_0)^2 - R^2 \right) (-3.0 + 2 y_0) \\
+ 2 \left( (230 - x_0)^2 + (0.6 - y_0)^2 - R^2 \right) (-1.2 + 2 y_0) \\
+ 2 \left( (240 - x_0)^2 + (0.3 - y_0)^2 - R^2 \right) (-0.6 + 2 y_0) = 0
\end{align*}\]
\[-4 ((130 - x0)^2 + (26.1 - y0)^2 - R^2) R - 4 ((140 - x0)^2 + (22.6 - y0)^2 - R^2) R \]
\[-4 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) R - 4 ((160 - x0)^2 + (14.9 - y0)^2 - R^2) R \]
\[-4 ((170 - x0)^2 + (11.8 - y0)^2 - R^2) R - 4 ((180 - x0)^2 + (9 - y0)^2 - R^2) R \]
\[-4 ((190 - x0)^2 + (6.2 - y0)^2 - R^2) R - 4 ((200 - x0)^2 + (4.1 - y0)^2 - R^2) R \]
\[-4 ((250 - x0)^2 + y0^2 - R^2) R = 0 \]

```plaintext
> sol := solve({eq1, eq2, eq3}, {x0, y0, R});

\[
\begin{align*}
\text{sol} &: = \{\text{sol} = \{y0 = 24.14512376 - 51.76271562 \text{i}, x0 = 159.8613219 + 164.5399717 \text{i}, R = 0\} , \cr
&\phantom{=} \{y0 = 24.14512376 + 51.76271562 \text{i}, x0 = 159.8613219 - 164.5399717 \text{i}, R = 0\} , \cr
&\phantom{=} \{x0 = 26.9168827 - 97.73889460 \text{i}, y0 = 152.4192633 - 30.27025113 \text{i}, R = 0\} , \cr
&\phantom{=} \{x0 = 26.9168827 + 97.73889460 \text{i}, y0 = 152.4192633 + 30.27025113 \text{i}, R = 0\} , \cr
&\phantom{=} \{x0 = 159.2001344, y0 = 41.52880421, R = 0\} , \cr
&\phantom{=} \{R = -348.3920217, x0 = 253.8662401, y0 = 350.3688105\} , \cr
&\phantom{=} \{x0 = 253.8662401, y0 = 350.3688105, R = 348.3920217\} \cr
\end{align*}
\]

\[
\circ := (x - x0)^2 + (y - y0)^2 = R^2
\]

```
$R = 348 \text{ m}$
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, ...

\begin{verbatim}
> restart;
> xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300];
> yy := [105.5, 83, 63.1, 46.7, 30.1, 18.7, 9.28, 3.0, 2.2, 7];
> n := 11;
\end{verbatim}

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \((x_i, y_i)\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\begin{verbatim}
> F := (x0, y0, R) -> sum((xx[i]-x0)^2+(yy[i]-y0)^2-R^2)^2, i=1..n; 
\end{verbatim}

\[ F : = (x0, y0, R) \rightarrow \sum_{i=1}^{n} ((xx_i - x0)^2 + (yy_i - y0)^2 - R^2)^2 \]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, \) and \(R\).

\begin{verbatim}
> eq1 := diff(F(x0, y0, R), x0)=0;
\end{verbatim}

\[ eq1 : = 4 (x0^2 + (105.5 - y0)^2 - R^2) x0 + 2 ((30 - x0)^2 + (83 - y0)^2 - R^2) (-60 + 2 x0) \]

\[ \quad \quad + 2 ((60 - x0)^2 + (63.1 - y0)^2 - R^2) (-120 + 2 x0) \]

\[ \quad \quad + 2 ((90 - x0)^2 + (46.7 - y0)^2 - R^2) (-180 + 2 x0) \]

\[ \quad \quad + 2 ((120 - x0)^2 + (30.1 - y0)^2 - R^2) (-240 + 2 x0) \]

\[ \quad \quad + 2 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) (-300 + 2 x0) \]

\[ \quad \quad + 2 ((180 - x0)^2 + (9 - y0)^2 - R^2) (-360 + 2 x0) \]

\[ \quad \quad + 2 ((210 - x0)^2 + (2.8 - y0)^2 - R^2) (-420 + 2 x0) \]

\[ \quad \quad + 2 ((240 - x0)^2 + (3 - y0)^2 - R^2) (-480 + 2 x0) \]

\[ \quad \quad + 2 ((270 - x0)^2 + (1.2 - y0)^2 - R^2) (-540 + 2 x0) \]

\[ \quad \quad + 2 ((300 - x0)^2 + (7 - y0)^2 - R^2) (-600 + 2 x0) = 0 \]

\begin{verbatim}
> eq2 := diff(F(x0, y0, R), y0)=0;
\end{verbatim}

\[ eq2 : = 2 (x0^2 + (105.5 - y0)^2 - R^2) (-211.0 + 2 y0) \]

\[ \quad \quad + 2 ((30 - x0)^2 + (83 - y0)^2 - R^2) (-166 + 2 y0) \]

\[ \quad \quad + 2 ((60 - x0)^2 + (63.1 - y0)^2 - R^2) (-126.2 + 2 y0) \]

\[ \quad \quad + 2 ((90 - x0)^2 + (46.7 - y0)^2 - R^2) (-93.4 + 2 y0) \]

\[ \quad \quad + 2 ((120 - x0)^2 + (30.1 - y0)^2 - R^2) (-60.2 + 2 y0) \]

\[ \quad \quad + 2 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) (-27.6 + 2 y0) \]

\[ \quad \quad + 2 ((180 - x0)^2 + (9 - y0)^2 - R^2) (-13.8 + 2 y0) \]

\[ \quad \quad + 2 ((210 - x0)^2 + (2.8 - y0)^2 - R^2) (-4.2 + 2 y0) \]

\[ \quad \quad + 2 ((240 - x0)^2 + (3 - y0)^2 - R^2) (-1.8 + 2 y0) \]

\[ \quad \quad + 2 ((270 - x0)^2 + (1.2 - y0)^2 - R^2) (-0.6 + 2 y0) \]

\[ \quad \quad + 2 ((300 - x0)^2 + (7 - y0)^2 - R^2) (0.6 + 2 y0) = 0 \]

\begin{verbatim}
> eq3 := diff(F(x0, y0, R), R)=0;
\end{verbatim}

\[ eq3 : = 2 (x0^2 + (105.5 - y0)^2 - R^2) (-9.225 + 2 R) \]

\[ \quad \quad + 2 ((30 - x0)^2 + (83 - y0)^2 - R^2) (-6.9 + 2 R) \]

\[ \quad \quad + 2 ((60 - x0)^2 + (63.1 - y0)^2 - R^2) (-3.44 + 2 R) \]

\[ \quad \quad + 2 ((90 - x0)^2 + (46.7 - y0)^2 - R^2) (-1.7 + 2 R) \]

\[ \quad \quad + 2 ((120 - x0)^2 + (30.1 - y0)^2 - R^2) (-0.86 + 2 R) \]

\[ \quad \quad + 2 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) (-0.43 + 2 R) \]

\[ \quad \quad + 2 ((180 - x0)^2 + (9 - y0)^2 - R^2) (-0.21 + 2 R) \]

\[ \quad \quad + 2 ((210 - x0)^2 + (2.8 - y0)^2 - R^2) (-0.105 + 2 R) \]

\[ \quad \quad + 2 ((240 - x0)^2 + (3 - y0)^2 - R^2) (-0.0525 + 2 R) \]

\[ \quad \quad + 2 ((270 - x0)^2 + (1.2 - y0)^2 - R^2) (-0.026 + 2 R) \]

\[ \quad \quad + 2 ((300 - x0)^2 + (7 - y0)^2 - R^2) (-0.013 + 2 R) = 0 \]
\[ + 2 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) (-37.4 + 2 y0) \\
+ 2 ((180 - x0)^2 + (9 - y0)^2 - R^2) (-18 + 2 y0) \\
+ 2 ((210 - x0)^2 + (2.8 - y0)^2 - R^2) (-5.6 + 2 y0) \\
+ 2 ((240 - x0)^2 + (.3 - y0)^2 - R^2) (-.6 + 2 y0) \\
+ 2 ((270 - x0)^2 + (.2 - y0)^2 - R^2) (-.4 + 2 y0) \\
+ 2 ((300 - x0)^2 + (7 - y0)^2 - R^2) (-14 + 2 y0) = 0 \]

```plaintext
> eq3 := diff(F(x0, y0, R), R) = 0;

\[ eq3 := -4 (x0^2 + (10.5 - y0)^2 - R^2) R - 4 ((30 - x0)^2 + (83 - y0)^2 - R^2) R \\
- 4 ((60 - x0)^2 + (63.1 - y0)^2 - R^2) R - 4 ((90 - x0)^2 + (46.7 - y0)^2 - R^2) R \\
- 4 ((120 - x0)^2 + (30.1 - y0)^2 - R^2) R - 4 ((150 - x0)^2 + (18.7 - y0)^2 - R^2) R \\
- 4 ((180 - x0)^2 + (9 - y0)^2 - R^2) R - 4 ((210 - x0)^2 + (2.8 - y0)^2 - R^2) R \\
- 4 ((240 - x0)^2 + (.3 - y0)^2 - R^2) R - 4 ((270 - x0)^2 + (.2 - y0)^2 - R^2) R \\
- 4 ((300 - x0)^2 + (7 - y0)^2 - R^2) R = 0 \]

> sol := solve([eq1, eq2, eq3], [x0, y0, R])

\[ sol := \{ y0 = 30.23911686 - 97.60827768 I, x0 = 142.9796088 - 33.42868907 I, R = 0 \}, \]
\[ y0 = 30.23911686 + 97.60827768 I, x0 = 142.9796088 + 33.42868907 I, R = 0 \}, \]
\[ x0 = 149.9073226 - 164.6452924 I, y0 = 27.44147698 + 57.12204900 I, R = 0 \}, \]
\[ y0 = 27.44147698 - 57.12204900 I, x0 = 149.9073226 + 164.6452924 I, R = 0 \}, \]
\[ y0 = 44.31206841, x0 = 149.9297309, R = 0 \}, \]
\[ R = -363.3261040, x0 = 258.7930754, y0 = 364.9027910 \}, \]
\[ x0 = 258.7930754, R = 363.3261040, y0 = 364.9027910 \}
```

> circ := (x-x0)^2 + (y-y0)^2 = R^2

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedralplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot ]

> pl := implicitplot(subs(sol[7], circ), x = 0 .. 400, y = -300 .. 300, numpoints = 4000):

> with(statplots);

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift]
\texttt{p2 := scatterplot(xx, yy, color=black);}
\texttt{display([p1, p2], scaling=constrained);}
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled.

\[
\begin{align*}
\text{restart:} \\
x &= [50, 100, 150, 200, 250] \\
xx &= [50, 100, 150, 200, 250] \\
y &= [68.8, 41.3, 18.7, 4.1, 0] \\
yy &= [68.8, 41.3, 18.7, 4.1, 0] \\
n &= 5 \\
R &= \sum_{i=1}^{n} ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2 \rightarrow \sum_{i=1}^{n} ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2 \\
F &= (x0, y0, R) \\
We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).
\]

\[
\begin{align*}
eq 1: & \quad \text{diff}(F(x0, y0, R), x0) = 0; \\
eq 1: & \quad f(x0, y0, R) = 0; \\
eq 1: & \quad (50-x0)^2 + (68.8-y0)^2 - R^2 \rightarrow (100+2x0) + 2 (100-x0)^2 + (41.3-y0)^2 - R^2 \rightarrow (200+2x0) + 2 (150-x0)^2 + (18.7-y0)^2 - R^2 \rightarrow (500+2x0) + 2 (200-x0)^2 + (4.1-y0)^2 - R^2 \rightarrow (400+2x0) + 2 (250-x0)^2 + y0^2 - R^2 \rightarrow (500+2x0) = 0 \\
eq 1: & \quad \text{diff}(F(x0, y0, R), y0) = 0; \\
eq 2: & \quad f(x0, y0, R) = 0; \\
eq 2: & \quad (50-x0)^2 + (68.8-y0)^2 - R^2 \rightarrow (100+2y0) + 2 (100-x0)^2 + (41.3-y0)^2 - R^2 \rightarrow (200+2y0) + 2 (150-x0)^2 + (18.7-y0)^2 - R^2 \rightarrow (370+2y0) + 2 (200-x0)^2 + (4.1-y0)^2 - R^2 \rightarrow (820+2y0) + 2 (250-x0)^2 + y0^2 - R^2 \rightarrow (500+2y0) = 0 \\
eq 3: & \quad \text{diff}(F(x0, y0, R), R) = 0; \\
eq 3: & \quad f(x0, y0, R) = 0; \\
eq 3: & \quad (50-x0)^2 + (68.8-y0)^2 - R^2 \rightarrow (100+2y0) + 4 (100-x0)^2 + (41.3-y0)^2 - R^2 \rightarrow (200+2y0) + 4 (150-x0)^2 + (18.7-y0)^2 - R^2 \rightarrow (370+2y0) + 4 (200-x0)^2 + (4.1-y0)^2 - R^2 \rightarrow (820+2y0) + 4 (250-x0)^2 + y0^2 - R^2 \rightarrow (500+2y0) = 0 \\
\text{sol:} & \quad \text{solve}\{(eq1, eq2, eq3), \{x0, y0, R\}\}; \\
sol: & \quad \{ R = 0, x0 = 146.6478212 - 25.13026781 I, y0 = 25.12078684 - 71.59862166 I \}, \\
\{ R = 0, y0 = 25.12078684 + 71.59862166 I, x0 = 146.6478212 + 25.13026781 I \}.
\]
circ := (x - xo)^2 + (y - yo)^2 = R^2

> with(plots):

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listconplot, listconplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedralplot, repplot, root locus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

> p1 := implicitplot(subs(sol[1], circ), x=0..400, y=-300..300, numpoints=4000):

> with(stats[statplots]):

[boxplot, histogram, scatterplot, xscale, xshift, yexchange, xexchange, yscale, yshift, yexchange, xscale, xshift]

> p2 := scatterplot(xx, yy, color=black):

> display({p1, p2}, scaling=constrained);
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The data I entered was labelled.

> restart;
> xx := [50, 150, 250];
> yy := [68.8, 18.7, 0];
> n := 3;

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1..n\), then the \(n\) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, \) and \(R\).

> eq1 := diff(P(x0, y0, R), x0) = 0;

\[
eq 1 := 2 ((50 - x_0)^2 + (68.8 - y_0)^2 - R^2) (-100 + 2 x_0)
\]

\[
+ 2 ((150 - x_0)^2 + (18.7 - y_0)^2 - R^2) (-300 + 2 x_0)
\]

\[
+ 2 ((250 - x_0)^2 + y_0^2 - R^2) (-500 + 2 x_0) = 0
\]

> eq2 := diff(P(x0, y0, R), y0) = 0;

\[
eq 2 := 2 ((50 - x_0)^2 + (68.8 - y_0)^2 - R^2) (-137.6 + 2 y_0)
\]

\[
+ 2 ((150 - x_0)^2 + (18.7 - y_0)^2 - R^2) (-37.4 + 2 y_0) + 4 ((250 - x_0)^2 + y_0^2 - R^2) y_0 = 0
\]

> eq3 := diff(P(x0, y0, R), R) = 0;

\[
eq 3 := -4 ((50 - x_0)^2 + (68.8 - y_0)^2 - R^2) R - 4 ((150 - x_0)^2 + (18.7 - y_0)^2 - R^2) R
\]

\[
- 4 ((250 - x_0)^2 + y_0^2 - R^2) R = 0
\]

> sol := solve({eq1, eq2, eq3}, {x0, y0, R});

\[
sol := \{ R = 0, x_0 = 146.7899989 + 28.53713094 I, y_0 = 27.74532828 - 82.56900615 I \},
\]

\[
\{ R = 0, x_0 = 27.74532828 + 82.56900615 I, y_0 = 146.7899989 + 28.53713094 I \},
\]

\[
\{ R = 0, x_0 = 149.9657160, y_0 = 34.24400678 \},
\]

\[
\{ R = 0, x_0 = 26.56134267 + 49.09184693 I, y_0 = 149.9939816 - 141.5322574 I \},
\]

\[
\{ R = 0, x_0 = 149.9939816 + 141.5322574 I, y_0 = 26.56134267 - 49.09184693 I \},
\]

\[
\{ x_0 = 269.8179389, y_0 = 382.7079618, R = -383.2207389 \},
\]

\[
\{ x_0 = 269.8179389, y_0 = 382.7079618, R = 383.2207389 \}
\]
\[ \text{circ} := (x-x0)^2 + (y-y0)^2 = R^2 \]

\[ \text{with(plots);} \]

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odepplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported, polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

\[ \text{p1:=implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);} \]

\[ \text{with(stats[statplots]);} \]

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift, yxexchange, zscale, zshift]

\[ \text{p2:=scatterplot(xx, yy, color=black);} \]

\[ \text{display([p1, p2], scaling\_constrained);} \]
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as \( xx \), and the list of y coordinates as \( yy \). The number of points in the list is \( n \).

The index of the loop is \( i \), which refers to the index of the first of the three points in the list. The output of the loop at each step is \( i \), and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative \( R \). If the three points are collinear, then no solution can be found, so the loop simply outputs the value of \( i \). The data I entered was labelled, "Futotch VIII, B4E".

```plaintext
> restart;
> xx := [10, 10, 20, 20, 30, 30, 40, 40, 50, 50, 60, 60, 70, 70, 80, 80, 90, 90, 100, 100, 110, 110, 120, 120, 130, 130, 140, 140, 150, 150, 160, 160, 170, 170, 180, 180, 190, 190, 200, 200, 210, 210, 220, 220, 230, 230, 240, 240, 250, 250, 260, 260, 270, 270, 280, 280, 290, 290, 300, 300, 310, 310];
> yy := [105.5, 97.2, 89.2, 83, 75.6, 68.8, 63.1, 57.2, 51.9, 46.7, 41.3, 35.5, 30.1, 26.1, 22.6, 18.7, 14.9, 11.8, 9.62, 4.28, 1.5, 1.5, 0.6, 0.3, 0.2, 0.2, 1, 5, 4.6, 7, 9.8, 10];
> cir := (x, y) -> ((x-x0)^2 + (y-y0)^2) = R^2;
> cir := (x, y) -> (x-x0)^2 + (y-y0)^2 = R^2;
> n := 33;
> for i from 1 to n-2 do
>   eq1 := cir(xx[i], yy[i]);
>   eq2 := cir(xx[i+1], yy[i+1]);
>   eq3 := cir(xx[i+2], yy[i+2]);
>   print(i, solve([eq1, eq2, eq3], [x0, y0, R]));
> od:
```

1, \( y0 = 652.0166667, x0 = 462.0533333, R = -715.6631539 \)
2, \( y0 = 652.0166667, x0 = 462.0533333, R = 715.6631539 \)
3, \( y0 = 173.2111111, x0 = 79.00888889, R = -102.6640191 \)
4, \( y0 = 173.2111111, x0 = 79.00888889, R = 102.6640191 \)
5, \( y0 = 326.4333333, x0 = -307.5031401, R = 217.8786667 \)
6, \( y0 = 326.4333333, x0 = -307.5031401, R = 217.8786667 \)
7, \( y0 = 195.4954545, x0 = 128.8409091, R = -149.2234135 \)
8, \( y0 = 195.4954545, x0 = 128.8409091, R = 149.2234135 \)
9, \( y0 = -327.5270000, x0 = -772.4864000, y0 = -605.5150000 \)
10, \( y0 = -327.5270000, x0 = -772.4864000, y0 = 605.5150000 \)
11, \( y0 = 276.2833333, x0 = 192.5186667, R = \)
8. \( \gamma = 1327.550000, x = 749.6900000, R = -1440.752449 \),
    \( \gamma = 1327.550000, R = 1440.752449, x = 749.6900000 \)
9. \( \gamma = -593.800000, x = -249.4120000, R = -724.8729239 \),
    \( \gamma = -593.800000, x = -249.4120000, R = 724.8729239 \)
10. \( \gamma = -287.200000, x = -83.8480000, R = -376.4469911 \),
    \( \gamma = -287.200000, x = -83.8480000, R = 376.4469911 \)
11. \( R = -376.4469911, y = 364.0, x = 293.8480000 \),
    \( R = 376.4469911, y = 364.0, x = 293.8480000 \)
12. \( R = -96.60620545, x = 160.8228571, y = 117.6571429 \),
    \( x = 160.8228571, y = 117.6571429, R = 96.60620545 \)
13. \( x = 215.500000, R = -243.7382048, y = 254.3500000 \),
    \( x = 215.500000, y = 254.3500000, R = 243.7382048 \)
14. \( y = -261.7250600, x = 34.87375000, R = -303.1373188 \),
    \( y = -261.7250600, x = 34.87375000, R = 303.1373188 \)
15. \( x = 592.0570000, y = 1166.9500000, R = -1230.403370 \),
    \( x = 592.0570000, y = 1166.9500000, R = 1230.403370 \)
16. \( x = 215.0915714, y = 174.9357143, R = -169.2528023 \),
    \( x = 215.0915714, y = 174.9357143, R = 169.2528023 \)
17. \( R = -377.8448785, x = 276.8686667, y = 374.2166667 \),
    \( R = 377.8448785, x = 276.8686667, y = 374.2166667 \)
18
19. \( x = 227.0580000, R = -156.0705646, y = 157.8071429 \),
    \( x = 227.0580000, y = 157.8071429, R = 156.0705646 \)
20. \( x = -221.8301250, y = 132.9125000, R = -130.6492040 \),
    \( x = 221.8301250, y = 132.9125000, R = 130.6492040 \)
21
22. \( x = 247.8217500, y = 254.6250000, R = -254.6493970 \),
    \( x = 247.8217500, y = 254.6250000, R = 254.6493970 \)
23. \( x = 240.0270000, y = 168.0166667, R = -167.7166668 \),
    \( R = 167.7166668, x = 240.0270000, y = 168.0166667 \)
24
25. \( x = 200.1300000, x = 250.9994000, R = -200.1324954 \),
    \( x = 200.1300000, x = 250.9994000, R = 200.1324954 \)
26. \( R = -500.1249944, y = -499.9000000, x = 265.0 \),
    \( R = 500.1249944, y = -499.9000000, x = 265.0 \)
27. \{ y_0 = 77.77307692, R = -77.73404829, x_0 = 265. \},
\{ y_0 = 77.77307692, R = 77.73404829, x_0 = 265. \}
28. \{ x_0 = 267.2852222, y_0 = 60.19444444, R = -60.05583554 \},
\{ x_0 = 267.2852222, R = 60.05583554, y_0 = 60.19444444 \}
29. \{ y_0 = -149.2357143, x_0 = 332.2085714, R = -159.5211287 \},
\{ y_0 = -149.2357143, x_0 = 332.2085714, R = 159.5211287 \}
30. \{ x_0 = 230.6320000, R = -275.8639509, y_0 = 274. \},
\{ x_0 = 230.6320000, R = 275.8639509, y_0 = 274. \}
31. \{ x_0 = 325.3000000, y_0 = -64.10000000, R = -75.46721142 \},
\{ x_0 = 325.3000000, y_0 = -64.10000000, R = 75.46721142 \}
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx and the list of y coordinates as yy. The number of points in the list is n. The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "Futrocks VIII, B4E".

```
> restart;
> xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300];
    xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300]
> yy := [105.5, 83, 63.1, 46.7, 30.1, 18.7, 9.2, 0, 0.3, 0.2, 7];
    yy := [105.5, 83, 63.1, 46.7, 30.1, 18.7, 9.2, 0, 0.3, 0.2, 7]
> cir := (x, y) -> ((x-x0)^2 + (y-y0)^2 = R^2);
    cir := (x, y) -> (x-x0)^2 + (y-y0)^2 = R^2
> n := 11;
    n := 11
> for i from 1 to n-2 do
    eq1 := cir(xx[i], yy[i]);
    eq2 := cir(xx[i+1], yy[i+1]);
    eq3 := cir(xx[i+2], yy[i+2]);
    print(i, solve({eq1, eq2, eq3}, {x0, y0, R}));
od:
1. {x0 = 602.6653846, y0 = 396.3115385, R = -635.7957653 },
    {y0 = 602.6653846, x0 = 396.3115385, R = 635.7957653}
2. {y0 = 415.2385714, x0 = 271.9850857, R = -411.0222014 },
    {y0 = 415.2385714, x0 = 271.9850857, R = 411.0222014}
3. {R = -6689.306503, x0 = -3133.660000, y0 = -5814.600000 },
    {R = 6689.306503, x0 = -3133.660000, y0 = -5814.600000 }
4. {y0 = 242.1692308, x0 = 217.7523077, R = -233.5141801 },
    {y0 = 242.1692308, x0 = 217.7523077, R = 233.5141801}
5. {x0 = 359.0513529, y0 = 614.0088235, R = -630.9477502 },
    {x0 = 359.0513529, y0 = 614.0088235, R = 630.9477502 }
6. {x0 = 252.6963143, y0 = 285.0757143, R = -285.4865218 },
    {x0 = 252.6963143, y0 = 285.0757143, R = 285.4865218}
7. {x0 = 245.8777027, y0 = 252.0824324, R = -251.8510287 },
    {x0 = 245.8777027, y0 = 252.0824324, R = 251.8510287}
8. {y0 = 376.6041667, x0 = 256.2545139, R = -376.6550611 },
    {R = 376.6550611, y0 = 376.6041667, x0 = 256.2545139}
9. {y0 = 133.9862319, x0 = 255.4457874, R = -134.5755585 }
```
{ x0 = 133.9862319, y0 = 255.4457874, R = 134.5755585 }
This worksheet is to find a circle which best fits a list of data points. The \( x \) coordinates are given as a list \( xx \), and the corresponding \( y \) coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled:

\[
yy := \{0.0, 0.0, 0.0, 0.1, 0.8, 1.3, 1.7, 3.4, 3.5, 7.7, 7.4, 9.4, 11.2, 13.7, 16.6, 19.4\}
\]

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \((x_i, y_i)\), \(i = 1 \ldots n\), then the \( n \) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\):

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} \left((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\right)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 1 := 2(-10 \cdot 2 \cdot x_0) + 2(100 - 2 \cdot x_0) + 2(30 \cdot 0 - 2 \cdot R^2) (-10 + 2 \cdot x_0)
\]

\[
+ 2(100 - 2 \cdot x_0) + 2(30 - 0 - 2 \cdot R^2) (-100 + 2 \cdot x_0)
\]

\[
+ 2(100 - 2 \cdot x_0) + 2(100 - 2 \cdot x_0) + 2(40 - 0 - 2 \cdot R^2) (-40 + 2 \cdot x_0).
\]
\[ + 2 ((165 - x0)^2 + (19.4 - y0)^2 - R^2) (-330 + 2 x0) = 0 \]

\[ eq2 := \text{diff}(F(x0, y0, R), y0) = 0; \]

\[ eq2 := 4 ((5 - x0)^2 + y0^2 - R^2) y0 + 4 ((10 - x0)^2 + y0^2 - R^2) y0 + 4 ((20 - x0)^2 + y0^2 - R^2) y0 + 4 ((30 - x0)^2 + y0^2 - R^2) y0 + 4 ((40 - x0)^2 + y0^2 - R^2) y0 + 2 ((50 - x0)^2 + (1.1 - y0)^2 - R^2) (-2 + 2 y0) + 2 ((60 - x0)^2 + (.8 - y0)^2 - R^2) (-1.6 + 2 y0) + 2 ((70 - x0)^2 + (1.3 - y0)^2 - R^2) (-2.6 + 2 y0) + 2 ((80 - x0)^2 + (1.7 - y0)^2 - R^2) (-3.4 + 2 y0) + 2 ((90 - x0)^2 + (3 - y0)^2 - R^2) (-6 + 2 y0) + 2 ((100 - x0)^2 + (4.3 - y0)^2 - R^2) (-8.6 + 2 y0) + 2 ((110 - x0)^2 + (5.7 - y0)^2 - R^2) (-11.4 + 2 y0) + 2 ((120 - x0)^2 + (7.4 - y0)^2 - R^2) (11.4 + 2 y0) + 2 ((130 - x0)^2 + (9.4 - y0)^2 - R^2) (18.8 + 2 y0) + 2 ((140 - x0)^2 + (11.2 - y0)^2 - R^2) (22.4 + 2 y0) + 2 ((150 - x0)^2 + (13.7 - y0)^2 - R^2) (27.4 + 2 y0) + 2 ((160 - x0)^2 + (16.6 - y0)^2 - R^2) (33.2 + 2 y0) + 2 ((165 - x0)^2 + (19.4 - y0)^2 - R^2) (-38.8 + 2 y0) = 0 \]

\[ eq3 := \text{diff}(F(x0, y0, R), R) = 0; \]

\[ eq3 := -4 ((5 - x0)^2 + y0^2 - R^2) R - 4 ((10 - x0)^2 + y0^2 - R^2) R - 4 ((20 - x0)^2 + y0^2 - R^2) R - 4 ((30 - x0)^2 + y0^2 - R^2) R - 4 ((40 - x0)^2 + y0^2 - R^2) R - 4 ((50 - x0)^2 + (1.1 - y0)^2 - R^2) R - 4 ((60 - x0)^2 + (.8 - y0)^2 - R^2) R - 4 ((70 - x0)^2 + (1.3 - y0)^2 - R^2) R - 4 ((80 - x0)^2 + (1.7 - y0)^2 - R^2) R - 4 ((90 - x0)^2 + (3 - y0)^2 - R^2) R - 4 ((100 - x0)^2 + (4.3 - y0)^2 - R^2) R - 4 ((110 - x0)^2 + (5.7 - y0)^2 - R^2) R - 4 ((120 - x0)^2 + (7.4 - y0)^2 - R^2) R - 4 ((130 - x0)^2 + (9.4 - y0)^2 - R^2) R - 4 ((140 - x0)^2 + (11.2 - y0)^2 - R^2) R - 4 ((150 - x0)^2 + (13.7 - y0)^2 - R^2) R - 4 ((160 - x0)^2 + (16.6 - y0)^2 - R^2) R - 4 ((165 - x0)^2 + (19.4 - y0)^2 - R^2) R = 0 \]

\[ sol := \text{solve}(\{eq1, eq2, eq3\}, \{x0, y0, R\}); \]

\[ sol := \{y0 = 4.211949372 + 9.760646614 \mathbb{R}, R = 0, x0 = 84.99641345 + 88.35538401 \mathbb{R}\}, \{y0 = 4.211949372 + 9.760646614 \mathbb{R}, x0 = 84.99641345 + 88.35538401 \mathbb{R}, R = 0\}, \{R = 85.00957410, y0 = 7.321051874\}, \{R = 0, x0 = 85.46604579 + 5.645822658 \mathbb{I}, y0 = 4.274775972 + 51.19423476 \mathbb{I}\}, \{R = 0, y0 = 4.274775972 + 51.19423476 \mathbb{I}, x0 = 85.46604579 + 5.645822658 \mathbb{I}\}, \{y0 = 461.1283442, R = -461.4712997, x0 = 35.01806212 \mathbb{I}\}, \{R = -461.4712997, y0 = 461.1283442, x0 = 35.01806212 \mathbb{I}\} \]

\[ circ := (x - x0)^2 + (y - y0)^2 = R^2 \]
with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported, polyhedrplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

> pl := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);

> with(stats[statplots]);

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift, yexchange, zscale, zshift]

> p2 := scatterplot(xx, yy, color=black);
> display([p1, p2], scaling=constrained);
R = 4.61 MV
This worksheet is to find a circle which best fits a list of data points. The \( x \) coordinates are given as a list \( xx \), and the corresponding \( y \) coordinates are given as a list \( yy \). The number of data points is \( n \). The data entered was labelled. 

\[
\begin{align*}
&\text{restart;} \\
&\text{x}:=[10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150]; \\
&xx:=[10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150] \\
&yy:=[4, 1.5, 1.5, 0.5, 0.2, 0.1, 0.8, 0.8, 1.4, 2.5, 3.8, 5.7, 2, 9.6, 13] \\
&n:=15;
\end{align*}
\]

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \( n \) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2
\]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\).

\[
\begin{align*}
eq 1 &= \text{diff}(F(x0, y0, R), x0) = 0; \\
eq 2 &= \text{diff}(F(x0, y0, R), y0) = 0;
\end{align*}
\]

\[
\begin{align*}
eq 1 &= 2((10 - x0)^2 + (4 - y0)^2 - R^2) (-20 + 2 x0) \\
&+ 2((20 - x0)^2 + (1.5 - y0)^2 - R^2) (-40 + 2 x0) \\
&+ 2((30 - x0)^2 + (1 - y0)^2 - R^2) (-60 + 2 x0) + 2((40 - x0)^2 + (5 - y0)^2 - R^2) (-80 + 2 x0) \\
&+ 2((50 - x0)^2 + (2 - y0)^2 - R^2) (-100 + 2 x0) \\
&+ 2((60 - x0)^2 + (1.5 - y0)^2 - R^2) (-120 + 2 x0) \\
&+ 2((70 - x0)^2 + (3 - y0)^2 - R^2) (-140 + 2 x0) \\
&+ 2((80 - x0)^2 + (3.5 - y0)^2 - R^2) (-160 + 2 x0) \\
&+ 2((90 - x0)^2 + (4 - y0)^2 - R^2) (-180 + 2 x0) \\
&+ 2((100 - x0)^2 + (4.5 - y0)^2 - R^2) (-200 + 2 x0) \\
&+ 2((110 - x0)^2 + (5.5 - y0)^2 - R^2) (-220 + 2 x0) \\
&+ 2((120 - x0)^2 + (6 - y0)^2 - R^2) (-240 + 2 x0) \\
&+ 2((130 - x0)^2 + (7 - y0)^2 - R^2) (-260 + 2 x0) \\
&+ 2((140 - x0)^2 + (8 - y0)^2 - R^2) (-280 + 2 x0) \\
&+ 2((150 - x0)^2 + (9 - y0)^2 - R^2) (-300 + 2 x0) = 0
\end{align*}
\]

\[
\begin{align*}
eq 2 &= 2((10 - x0)^2 + (4 - y0)^2 - R^2) (-8 + 2 x0); \\
\end{align*}
\]
\[ \begin{align*}
-2((20-x)^2+(1.5-y)^2-R^2)(-3.0-2y)+2((30-x)^2+(1-y)^2-R^2)(-2+2y)
&+2((40-x)^2+(y-0)^2-R^2)(-1.0+2y)+2((50-x)^2+(2-y)^2-R^2)(-4+2y)
&-2((60-x)^2+(1-y)^2-R^2)(-2+2y)+2((70-x)^2+(y-0)^2-R^2)(-1.6+2y)
&+2((80-x)^2+(y-0)^2-R^2)(-1+2y)
&+2((90-x)^2+(1.4-y)^2-R^2)(-2.8+2y)
&+2((100-x)^2+(2.5-y)^2-R^2)(-5.0+2y)
&+2((110-x)^2+(1.8-y)^2-R^2)(-7.6+2y)
&+2((120-x)^2+(5-y)^2-R^2)(-10+2y)
&+2((130-x)^2+(5.2-y)^2-R^2)(-14.4+2y)
&+2((140-x)^2+(9.6-y)^2-R^2)(-19.2+2y)
&+2((150-x)^2+(13-y)^2-R^2)(-26+2y) = 0
\end{align*} \]

> \texttt{eq3 := \text{diff}(R(x0,y0,R),R) = 0;}

\[ \begin{align*}
\text{solve(} \{ \text{eq1, eq2, eq3}, \{x0,y0,R \}\};
\end{align*} \]

> \texttt{circ := (x-x0)^2+(y-y0)^2 = R^2;}

> \texttt{with(plot3d);}
pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot

> p1 := implicitplot(subs(sol[1], circ), x=-400..400, y=-300..300, num_points=4000):
>
> with(stats[statplots]);
[ boxplot, histogram, scatterplot, tscale, xshift, xyexchange, xexchange, yscale, yshift,
  yxexchange, zscale, zshift ]
> p2 := scatterplot(xx, yy, color=black):
> display([p1, p2], scaling=constrained);
B5E
Port side
Futtock VII
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, \( \texttt{restart} \):

\[
xx := [3, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 316];
\]

\[
xx := [3, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 316];
\]

\[
yy := [6.2, 4.8, 5.2, 6.2, 2.1, 3.0, 8.0, 7.0, 6.0, 6.0, 4.0, 1.0, 0.1, 0.6, 1.0, 2, 3.1, 4.8, 6.4, 8.8, 10.6, 13.3; 6.2, 18.8, 22.5, 25.8, 28.2, 33.5, 36.7, 40.6, 44.5, 47.5];
\]

\[
n := 33;
\]

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0,\) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\):

\[
F := (x_0, y_0, R) \to \sum_{i=1}^{n} ((x_i-x_0)^2 + (y_i-y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]

\[
eq 3 := \text{diff}(F(x_0, y_0, R), R) = 0;
\]

\[
eq 1 := 2 ((190 - x_0)^2 + (6.4 - y_0)^2 - R^2) (-380 + 2 x_0)
+ 2 ((10 - x_0)^2 + (4.8 - y_0)^2 - R^2) (-20 + 2 x_0)
+ 2 ((40 - x_0)^2 + (2.2 - y_0)^2 - R^2) (-80 + 2 x_0)
+ 2 ((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) (-100 + 2 x_0)
+ 2 ((60 - x_0)^2 + (0.8 - y_0)^2 - R^2) (-120 + 2 x_0)
+ 2 ((20 - x_0)^2 + (4.5 - y_0)^2 - R^2) (-40 + 2 x_0)
+ 2 ((10 - x_0)^2 + (2.6 - y_0)^2 - R^2) (-60 + 2 x_0)
+ 2 ((160 - x_0)^2 + (2 - y_0)^2 - R^2) (-320 + 2 x_0)
+ 2 ((140 - x_0)^2 + (0.6 - y_0)^2 - R^2) (-280 + 2 x_0)
+ 2 ((150 - x_0)^2 + (1 - y_0)^2 - R^2) (-300 + 2 x_0) + 2 ((120 - x_0)^2 + y_0^2 - R^2) (-240 + 2 x_0)
+ 2 ((130 - x_0)^2 + (1 - y_0)^2 - R^2) (-260 + 2 x_0)
\]

\[
eq 2 := 2 ((190 - x_0)^2 + (6.4 - y_0)^2 - R^2) (-380 + 2 x_0)
+ 2 ((10 - x_0)^2 + (4.8 - y_0)^2 - R^2) (-20 + 2 x_0)
+ 2 ((40 - x_0)^2 + (2.2 - y_0)^2 - R^2) (-80 + 2 x_0)
+ 2 ((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) (-100 + 2 x_0)
+ 2 ((60 - x_0)^2 + (0.8 - y_0)^2 - R^2) (-120 + 2 x_0)
+ 2 ((20 - x_0)^2 + (4.5 - y_0)^2 - R^2) (-40 + 2 x_0)
+ 2 ((10 - x_0)^2 + (2.6 - y_0)^2 - R^2) (-60 + 2 x_0)
+ 2 ((160 - x_0)^2 + (2 - y_0)^2 - R^2) (-320 + 2 x_0)
+ 2 ((140 - x_0)^2 + (0.6 - y_0)^2 - R^2) (-280 + 2 x_0)
+ 2 ((150 - x_0)^2 + (1 - y_0)^2 - R^2) (-300 + 2 x_0) + 2 ((120 - x_0)^2 + y_0^2 - R^2) (-240 + 2 x_0)
+ 2 ((130 - x_0)^2 + (1 - y_0)^2 - R^2) (-260 + 2 x_0)
\]

\[
eq 3 := 2 ((190 - x_0)^2 + (6.4 - y_0)^2 - R^2) (-380 + 2 x_0)
+ 2 ((10 - x_0)^2 + (4.8 - y_0)^2 - R^2) (-20 + 2 x_0)
+ 2 ((40 - x_0)^2 + (2.2 - y_0)^2 - R^2) (-80 + 2 x_0)
+ 2 ((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) (-100 + 2 x_0)
+ 2 ((60 - x_0)^2 + (0.8 - y_0)^2 - R^2) (-120 + 2 x_0)
+ 2 ((20 - x_0)^2 + (4.5 - y_0)^2 - R^2) (-40 + 2 x_0)
+ 2 ((10 - x_0)^2 + (2.6 - y_0)^2 - R^2) (-60 + 2 x_0)
+ 2 ((160 - x_0)^2 + (2 - y_0)^2 - R^2) (-320 + 2 x_0)
+ 2 ((140 - x_0)^2 + (0.6 - y_0)^2 - R^2) (-280 + 2 x_0)
+ 2 ((150 - x_0)^2 + (1 - y_0)^2 - R^2) (-300 + 2 x_0) + 2 ((120 - x_0)^2 + y_0^2 - R^2) (-240 + 2 x_0)
+ 2 ((130 - x_0)^2 + (1 - y_0)^2 - R^2) (-260 + 2 x_0)
\]

\[
eq 3 := 2 ((190 - x_0)^2 + (6.4 - y_0)^2 - R^2) (-380 + 2 x_0)
+ 2 ((10 - x_0)^2 + (4.8 - y_0)^2 - R^2) (-20 + 2 x_0)
+ 2 ((40 - x_0)^2 + (2.2 - y_0)^2 - R^2) (-80 + 2 x_0)
+ 2 ((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) (-100 + 2 x_0)
+ 2 ((60 - x_0)^2 + (0.8 - y_0)^2 - R^2) (-120 + 2 x_0)
+ 2 ((20 - x_0)^2 + (4.5 - y_0)^2 - R^2) (-40 + 2 x_0)
+ 2 ((10 - x_0)^2 + (2.6 - y_0)^2 - R^2) (-60 + 2 x_0)
+ 2 ((160 - x_0)^2 + (2 - y_0)^2 - R^2) (-320 + 2 x_0)
+ 2 ((140 - x_0)^2 + (0.6 - y_0)^2 - R^2) (-280 + 2 x_0)
+ 2 ((150 - x_0)^2 + (1 - y_0)^2 - R^2) (-300 + 2 x_0) + 2 ((120 - x_0)^2 + y_0^2 - R^2) (-240 + 2 x_0)
+ 2 ((130 - x_0)^2 + (1 - y_0)^2 - R^2) (-260 + 2 x_0)
\]
\[ +2 ( (110 - x0)^2 + (y0 - y)^2 - R^2 ) (-220 + 2 x0) + 2 ( (70 - x0)^2 + (y0 - y)^2 - R^2 ) (-140 + 2 x0) + 2 ( (80 - x0)^2 + (y0 - y)^2 - R^2 ) (-160 + 2 x0) + 2 ( (90 - x0)^2 + (y0 - y)^2 - R^2 ) (-180 + 2 x0) + 2 ( (3 - x0)^2 + (6.2 - y0)^2 - R^2 ) (-6 + 2 x0) + 2 ( (100 - x0)^2 + (y0 - y)^2 - R^2 ) (-200 + 2 x0) + 2 ( (270 - x0)^2 + (y0 - y)^2 - R^2 ) (-540 + 2 x0) + 2 ( (260 - x0)^2 + (28.2 - y0)^2 - R^2 ) (-520 + 2 x0) + 2 ( (250 - x0)^2 + (22.5 - y0)^2 - R^2 ) (-500 + 2 x0) + 2 ( (240 - x0)^2 + (18.8 - y0)^2 - R^2 ) (-480 + 2 x0) + 2 ( (300 - x0)^2 + (40.6 - y0)^2 - R^2 ) (-600 + 2 x0) + 2 ( (290 - x0)^2 + (36.7 - y0)^2 - R^2 ) (-580 + 2 x0) + 2 ( (280 - x0)^2 + (33.5 - y0)^2 - R^2 ) (-560 + 2 x0) + 2 ( (230 - x0)^2 + (16.2 - y0)^2 - R^2 ) (-460 + 2 x0) + 2 ( (316 - x0)^2 + (47.5 - y0)^2 - R^2 ) (-632 + 2 x0) + 2 ( (310 - x0)^2 + (44.5 - y0)^2 - R^2 ) (-620 + 2 x0) + 2 ( (170 - x0)^2 + (3.1 - y0)^2 - R^2 ) (-340 + 2 x0) + 2 ( (180 - x0)^2 + (4.8 - y0)^2 - R^2 ) (-360 + 2 x0) + 2 ( (220 - x0)^2 + (13.3 - y0)^2 - R^2 ) (-440 + 2 x0) + 2 ( (210 - x0)^2 + (10.6 - y0)^2 - R^2 ) (-420 + 2 x0) + 2 ( (200 - x0)^2 + (8.8 - y0)^2 - R^2 ) (-400 + 2 x0) = 0 \]

\[ eq2 := \text{diff}(F(x0, y0, R), y0) = 0; \]

\[ eq2 := 2 ( (3 - x0)^2 + (6.2 - y0)^2 - R^2 ) (-12.4 + 2 y0) + 2 ( (10 - x0)^2 + (4.8 - y0)^2 - R^2 ) (-9.6 + 2 y0) + 2 ( (80 - x0)^2 + (4.8 - y0)^2 - R^2 ) (-1.2 + 2 y0) + 2 ( (90 - x0)^2 + (4.8 - y0)^2 - R^2 ) (-1.2 + 2 y0) + 2 ( (100 - x0)^2 + (4.8 - y0)^2 - R^2 ) (-0.8 + 2 y0) + 2 ( (210 - x0)^2 + (10.6 - y0)^2 - R^2 ) (-21.2 + 2 y0) + 2 ( (220 - x0)^2 + (13.3 - y0)^2 - R^2 ) (-26.6 + 2 y0) + 2 ( (70 - x0)^2 + (7.7 - y0)^2 - R^2 ) (-1.4 + 2 y0) + 2 ( (180 - x0)^2 + (4.8 - y0)^2 - R^2 ) (-9.6 + 2 y0) + 2 ( (190 - x0)^2 + (6.4 - y0)^2 - R^2 ) (-12.8 + 2 y0) + 2 ( (200 - x0)^2 + (8.8 - y0)^2 - R^2 ) (-17.6 + 2 y0) + 2 ( (310 - x0)^2 + (44.5 - y0)^2 - R^2 ) (-89.0 + 2 y0) \]
\[ +2 \left( (316 - x0)^2 + (47.5 - y0)^2 - R^2 \right) (-95.0 + 2y0) \\
+2 \left( (230 - x0)^2 + (16.2 - y0)^2 - R^2 \right) (-32.4 + 2y0) \\
+2 \left( (170 - x0)^2 + (3.1 - y0)^2 - R^2 \right) (-6.2 + 2y0) \\
+2 \left( (290 - x0)^2 + (36.7 - y0)^2 - R^2 \right) (-73.4 + 2y0) \\
+2 \left( (300 - x0)^2 + (40.6 - y0)^2 - R^2 \right) (-81.2 + 2y0) \\
+2 \left( (280 - x0)^2 + (33.5 - y0)^2 - R^2 \right) (-67.0 + 2y0) \\
+2 \left( (250 - x0)^2 + (22.5 - y0)^2 - R^2 \right) (-45.0 + 2y0) \\
+2 \left( (260 - x0)^2 + (25.8 - y0)^2 - R^2 \right) (-51.6 + 2y0) \\
+2 \left( (270 - x0)^2 + (28.2 - y0)^2 - R^2 \right) (-56.4 + 2y0) \\
+2 \left( (20 - x0)^2 + (4.5 - y0)^2 - R^2 \right) (-9.0 + 2y0) \\
+2 \left( (30 - x0)^2 + (2.6 - y0)^2 - R^2 \right) (-5.2 + 2y0) \\
+2 \left( (40 - x0)^2 + (2.2 - y0)^2 - R^2 \right) (-4.4 + 2y0) \\
+2 \left( (50 - x0)^2 + (1.3 - y0)^2 - R^2 \right) (-2.6 + 2y0) \\
+2 \left( (60 - x0)^2 + (8.0 - y0)^2 - R^2 \right) (-1.6 + 2y0) + 2 \left( (160 - x0)^2 + (2 - y0)^2 - R^2 \right) (-4 + 2y0) \\
+4 \left( (120 - x0)^2 + (0.2 - y0)^2 - R^2 \right) y0 + 2 \left( (130 - x0)^2 + (1 - y0)^2 - R^2 \right) (-2 + 2y0) \\
+4 \left( (140 - x0)^2 + (1.0 - y0)^2 - R^2 \right) (-1.2 + 2y0) \\
+2 \left( (150 - x0)^2 + (1 - y0)^2 - R^2 \right) (-2 + 2y0) + 2 \left( (110 - x0)^2 + (0.1 - y0)^2 - R^2 \right) (-2 + 2y0) \\
+2 \left( (240 - x0)^2 + (18.8 - y0)^2 - R^2 \right) (-37.6 + 2y0) = 0 \\
\]
\[-4 ((60 - x0)^2 + (8 - y0)^2 - R^2) - 4 ((20 - x0)^2 + (4.5 - y0)^2 - R^2) - 4 ((30 - x0)^2 + (2.6 - y0)^2 - R^2) - 4 ((40 - x0)^2 + (2.2 - y0)^2 - R^2) - 4 ((50 - x0)^2 + (1.3 - y0)^2 - R^2) = 0\]

\[\text{sol} := \text{solve}\{\text{eq1, eq2, eq3}\}, (x0, y0, R)\} ;\]
\[\text{sol} := \{x0 = 159.9577779, y0 = 18.73435959, R = 0\},\]
\[\{y0 = 8.238232444 - 20.98414209 I, x0 = 159.9900766 - 164.5283161 I, R = 0\},\]
\[\{x0 = 159.9900766 + 164.5283161 I, y0 = 8.238232444 + 20.98414209 I, R = 0\},\]
\[\{x0 = 161.7015955 - 12.18393973 I, y0 = 8.570537777 + 96.03790806 I, R = 0\},\]
\[\{x0 = 8.570537777 - 96.03790806 I, x0 = 161.7015955 + 12.18393973 I, R = 0\},\]
\[\{x0 = -8.19891710, y0 = 516.3347410, R = -517.4106330\},\]
\[\{x0 = 8.19891710, y0 = 516.3347410, R = 517.4106330\}\]
\[\text{circ} := (x - x0)^2 + (y - y0)^2 = R^2\]
\[\text{with(plots)} ;\]

\[\text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,}\]
\[\text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,}\]
\[\text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,}\]
\[\text{listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,}\]
\[\text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_s-supported,}\]
\[\text{polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,}\]
\[\text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot}\]
\[\text{p1 := implicitplot(subs(sol[7], circ), x = 0..400, y = -300..300, numpoints = 4000)} ;\]
\[\text{with(stats[statplots])} ;\]

\[\text{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,}\]
\[\text{yexchange, zscale, zshift}\]
\[\text{p2 := scatterplot(x, y, color = black);}\]
\[\text{display([p1, p2], scaling = constrained);}\]
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The data I entered was labelled:

```
> restart;
> xx:=[3, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300];
> yy:=[6.2, 2.6, .8, .6, 0, 1, 4.8, 10.6, 18.8, 28.2, 40.6];
> n:=11;
```

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \text{ and } R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x[i]-x_0)^2 + (y[i]-y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, \text{ and } R\).

```
> eq1 := diff(F(x0, y0, R), x0) = 0;
> eq2 := diff(F(x0, y0, R), y0) = 0;
```

We aim to find the critical points by solving these equations.
\[ + 2 \left( (180 - x0)^2 + (4.8 - y0)^2 - R^2 \right) (-9.6 + 2 y0) \]
\[ + 2 \left( (210 - x0)^2 + (10.6 - y0)^2 - R^2 \right) (-21.2 + 2 y0) \]
\[ + 2 \left( (240 - x0)^2 + (18.8 - y0)^2 - R^2 \right) (-37.6 + 2 y0) \]
\[ + 2 \left( (270 - x0)^2 + (28.2 - y0)^2 - R^2 \right) (-56.4 + 2 y0) \]
\[ + 2 \left( (300 - x0)^2 + (40.6 - y0)^2 - R^2 \right) (-81.2 + 2 y0) = 0 \]

\[ \text{eq3 := diff(f(x0,x0,y0,R),R) = 0;} \]

\[ \text{eq3 := -4 \left( (3 - x0)^2 + (6.2 - y0)^2 - R^2 \right) R - 4 \left( (30 - x0)^2 + (2.6 - y0)^2 - R^2 \right) R} \]
\[ - 4 \left( (60 - x0)^2 + (8 - y0)^2 - R^2 \right) R - 4 \left( (90 - x0)^2 + (1.6 - y0)^2 - R^2 \right) R} \]
\[ - 4 \left( (120 - x0)^2 + (4.8 - y0)^2 - R^2 \right) R - 4 \left( (150 - x0)^2 + (1 - y0)^2 - R^2 \right) R} \]
\[ - 4 \left( (180 - x0)^2 + (4.8 - y0)^2 - R^2 \right) R - 4 \left( (210 - x0)^2 + (10.6 - y0)^2 - R^2 \right) R} \]
\[ - 4 \left( (240 - x0)^2 + (18.8 - y0)^2 - R^2 \right) R - 4 \left( (270 - x0)^2 + (28.2 - y0)^2 - R^2 \right) R} \]
\[ - 4 \left( (300 - x0)^2 + (40.6 - y0)^2 - R^2 \right) R = 0 \]

\[ \text{sol := solve([eq1, eq2, eq3], [x0, y0, R]);} \]

\[ \text{sol := \{R = 0, x0 = 150.0602555 - 163.7792652 I, y0 = 6.963176076 - 17.82088528 I\},} \]
\[ \{R = 0, x0 = 6.963176076 + 17.82088528 I, y0 = 150.0602555 + 163.7792652 I\},} \]
\[ \{R = 0, x0 = 7.315901971 - 95.54144427 I, y0 = 152.3750282 + 10.31258481 I\},} \]
\[ \{R = 0, x0 = 7.315901971 + 95.54144427 I, y0 = 152.3750282 - 10.31258481 I\},} \]
\[ \{R = 0, y0 = 17.01772753, x0 = 150.7196691\},} \]
\[ \{R = -525.0778296, x0 = 96.26747371, y0 = 523.9070268\},} \]
\[ \{R = 525.0778296, x0 = 96.26747371, y0 = 523.9070268\} \]

\[ \text{circ := (x-x0)^2 + (y-y0)^2 = R^2;} \]

\[ \text{with(plots);} \]

\[ \text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,} \]
\[ \text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,} \]
\[ \text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,} \]
\[ \text{listcontplot3d, listdENSITYplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,} \]
\[ \text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,} \]
\[ \text{polyhedraplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,} \]
\[ \text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot} \]

\[ \text{pl := implicitplot(subs(sol[7],circ),x=0..400,y=-300..300, numpoints=4000);} \]

\[ \text{with(stats[statplots]);} \]

\[ \text{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,} \]
\[ \text{yexchange, zscale, zshift} \]

\[ \text{p2 := scatterplot(xx,yy,color=black);} \]
> display([p1, p2], scaling=constrained);

R = 5.25 m
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter
the list of \(x\) coordinates as \(xx\), and the list of \(y\) coordinates as \(yy\). The number of points in the list is \(n\).
The index of the loop is \(i\), which refers to the index of the first of the three points in the list. The output
of the loop at each step is \(i\), and then the center and radius of the circle (unfortunately, not always in
the same order). There is always a spurious solution corresponding to a negative \(R\). If the three points
are colinear, then no solution can be found, so the loop simply outputs the value of \(i\). The data I entered
was labelled, "Futtwick VII, B5E".

```plaintext
> restart;

> xx := [3, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 316];

> yy := [6.2, 4.8, 4.5, 2.6, 2.2, 2.1, 3, 0.8, 0.7, 0.6, 0.5, 0.4, 0.1, 0.1, 0.6, 1.2, 3.1, 4.8, 6.4, 8.8, 10.6, 13.3, 16.2, 18.8, 22.5, 25.8, 28.2, 33.5, 36.7, 40.6, 44.5, 47.5];

> n := 33;

> for i from 1 to n-2 do

> eq1 := cir(xx[i], yy[i]);

> eq2 := cir(xx[i+1], yy[i+1]);

> eq3 := cir(xx[i+2], yy[i+2]);

> print(i, solve([eq1, eq2, eq3], [x0, y0, R]));
```

\[\begin{align*}
1. \{ & x0 = 16.53000000, y0 = 55.65000000, R = -51.26756675 \}. \\
& \{ x0 = 16.53000000, y0 = 55.65000000, R = 51.26756675 \} \\
2. \{ & x0 = 13.08581250, y0 = -59.15625000, R = -64.03065010 \}. \\
& \{ x0 = 13.08581250, y0 = -59.15625000, R = 64.03065010 \} \\
3. \{ & y0 = 70.52333333, R = -68.36120103, x0 = 37.72493333 \}. \\
& \{ y0 = 70.52333333, R = 68.36120103, x0 = 37.72493333 \} \\
4. \{ & y0 = -198.77000000, x0 = 26.95320000, R = -201.3930483 \}. \\
& \{ y0 = -198.77000000, x0 = 26.95320000, R = 201.3930483 \} \\
5. \{ & x0 = 67.57875000, y0 = 252.62500000, R = -251.9390166 \}. \\
& \{ x0 = 67.57875000, y0 = 252.62500000, R = 251.9390166 \} \\
6. \{ & y0 = -251.12500000, x0 = 67.50375000, R = -250.4374411 \}. \\
& \{ y0 = -251.12500000, x0 = 67.50375000, R = 250.4374411 \}
\end{align*}\]
8. \( x_0 = 1000.650000, R = -1000.062499, x_0 = 85. \)
   \( R = 1000.062499, y_0 = 1000.650000, x_0 = 85. \)
9. \( x_0 = 85., y_0 = -499.500000, R = -500.1249944 \)
   \( x_0 = 85., y_0 = -499.500000, R = 500.1249944 \)
10. \( x_0 = 74.98500000, R = -1000.962623, y_0 = -1000.250000 \)
    \( x_0 = 74.98500000, y_0 = -1000.250000, R = 1000.962623 \)
11. \( x_0 = 120.00300000, y_0 = 500.3500000, R = -500.3500000 \)
    \( x_0 = 120.00300000, y_0 = 500.3500000, R = 500.3500000 \)
12. \( R = 500.0500000, y_0 = 500.0500000, x_0 = 120. \)
    \( y_0 = 500.0500000, x_0 = 120., R = -500.0500000 \)
13. \( x_0 = 122.4962500, y_0 = 250.4250000, R = -250.4374411 \)
    \( x_0 = 122.4962500, y_0 = 250.4250000, R = 250.4374411 \)
14. \( x_0 = 185.0900000, y_0 = -1001.450000, R = -1003.063961 \)
    \( x_0 = 185.0900000, y_0 = -1001.450000, R = 1003.063961 \)
15. \( R = -168.0420680, x_0 = 138.2866667, y_0 = 168.6333333 \)
    \( x_0 = 138.2866667, y_0 = 168.6333333, R = 168.0420680 \)
16. \( y_0 = 1013.050000, x_0 = 53.8450000, R = -1016.607587 \)
    \( y_0 = 1013.050000, x_0 = 53.8450000, R = 1016.607587 \)
17. \( y_0 = 173.1833333, x_0 = 146.2303333, R = -171.7362435 \)
    \( y_0 = 173.1833333, x_0 = 146.2303333, R = 171.7362435 \)
18. \( x_0 = 349.4880000, y_0 = -1022.450000, R = -1041.138197 \)
    \( x_0 = 349.4880000, y_0 = -1022.450000, R = 1041.138197 \)
19. \( R = -132.7628020, y_0 = 136.6000000, x_0 = 164.0400000 \)
    \( R = 132.7628020, y_0 = 136.6000000, x_0 = 164.0400000 \)
20. \( x_0 = 236.5120000, y_0 = -165.3666667, R = -177.9526733 \)
    \( x_0 = 236.5120000, y_0 = -165.3666667, R = 177.9526733 \)
21. \( R = -119.8629540, x_0 = 183.7850000, y_0 = 127.5611111 \)
    \( R = 119.8629540, x_0 = 183.7850000, y_0 = 127.5611111 \)
22. \( y_0 = -559.9822111, x_0 = 552.5500000, y_0 = 69.0380000 \)
    \( y_0 = -559.9822111, x_0 = 552.5500000, y_0 = 69.0380000 \)
23. \( x_0 = 328.5783333, y_0 = -342.4166667, R = -371.9188102 \)
    \( x_0 = 328.5783333, y_0 = -342.4166667, R = 371.9188102 \)
24. \( R = -105.0064628, x_0 = 208.6088182, y_0 = 119.0045455 \)
    \( x_0 = 208.6088182, y_0 = 119.0045455, R = 105.0064628 \)
25. \( R = -297.3996879, y_0 = -258.2250000, x_0 = 348.1837500 \)
    \( R = 297.3996879, y_0 = -258.2250000, x_0 = 348.1837500 \)
26. \( \{ x0 = 294.1746667, y0 = -94.56111111, R = -125.1187632 \} \)

27. \( \{ x0 = 255.0354483, y0 = 68.51896552, R = -43.00647380 \} \)

28. \( \{ y0 = -23.24523810, R = -61.48425400, x0 = 303.6704762 \} \)

29. \( \{ x0 = 232.9565714, y0 = 197.7357143, R = -170.8404344 \} \)

30. \( \{ x0 = 270.5204545, y0 = 130.9590909, R = -95.04635138 \} \)

31. \( \{ x0 = 270.5204545, y0 = 130.9590909, R = -95.04635138 \} \)
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The data I entered was labelled.

```maple
restart;
xx := [50, 100, 150, 200, 250];
xx := [50, 100, 150, 200, 250]
yy := [1.3, .4, 1, 8.8, 22.5];
yy := [1.3, .4, 1, 8.8, 22.5]
n := 5;
```

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

```maple
eq1 := diff(F(x0, y0, R), x0) = 0;
eq1 := 2 ((50 - x0)^2 + (1.3 - y0)^2 - R^2) (-100 + 2 x0) + 2 ((100 - x0)^2 + (.4 - y0)^2 - R^2) (-200 + 2 x0) + 2 ((150 - x0)^2 + (1 - y0)^2 - R^2) (-300 + 2 x0) + 2 ((200 - x0)^2 + (8.8 - y0)^2 - R^2) (-400 + 2 x0) + 2 ((250 - x0)^2 + (22.5 - y0)^2 - R^2) (-500 + 2 x0) = 0
eq2 := diff(F(x0, y0, R), y0) = 0;
eq2 := 2 ((50 - x0)^2 + (1.3 - y0)^2 - R^2) (-2.6 + 2 y0) + 2 ((100 - x0)^2 + (.4 - y0)^2 - R^2) (-8 + 2 y0) + 2 ((150 - x0)^2 + (1 - y0)^2 - R^2) (-2 + 2 y0) + 2 ((200 - x0)^2 + (8.8 - y0)^2 - R^2) (-17.6 + 2 y0) + 2 ((250 - x0)^2 + (22.5 - y0)^2 - R^2) (-45.0 + 2 y0) = 0

\>
eq3 := diff(F(x0, y0, R), R) = 0;
eq3 := -4 ((50 - x0)^2 + (1.3 - y0)^2 - R^2) R - 4 ((100 - x0)^2 + (.4 - y0)^2 - R^2) R - 4 ((150 - x0)^2 + (1 - y0)^2 - R^2) R - 4 ((200 - x0)^2 + (8.8 - y0)^2 - R^2) R - 4 ((250 - x0)^2 + (22.5 - y0)^2 - R^2) R = 0

\>
sol := solve({eq1, eq2, eq3}, \{x0, y0, R\});
sol := \{R = 0, x0 = 149.9934950 - 22.5628370 i, y0 = 4.966579793 - 12.50916802 i\},
```

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\[ R = 0, \ x_0 = 149.9934950 + 122.5628370 \ i, \ y_0 = 4.966579793 + 12.50916802 \ i \].
\[ R = 0, \ y_0 = 10.40320535, \ x_0 = 150.0195135 \],
\[ R = 0, \ x_0 = 150.7597274 - 7.245292095 \ i, \ y_0 = 5.076824900 + 71.18320841 \ i \],
\[ R = 0, \ y_0 = 5.076824900 - 71.18320841 \ i, \ x_0 = 150.7597274 + 7.245292095 \ i \],
\[ y_0 = 480.9406945, \ R = -481.8330879, \ x_0 = 102.2228254 \],
\[ y_0 = 480.9406945, \ x_0 = 102.2228254, \ R = 481.8330879 \]

> circ := (x-x0)^2 + (y-y0)^2 = R^2

> with(plots);

> animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedralplot3d, polyhedralplot,
replot, rootplot, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot

> pl := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);

> with(stats[statplots]);

> boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,

> yexchange, zscale, zshift

> p2 := scatterplot(xx, yy, color=black);

> display([pl, p2], scaling=constrained);
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, \( xx := \{50, 150, 250\} \), \( yy := \{1.3, 1, 22.5\} \), \( n := 3 \).

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly though the given points \((x_i, y_i)\), \( i = 1 \ldots n \), then the \( n \) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\), and \( R \) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[
\begin{align*}
F := (x_0, y_0, R) & \rightarrow \sum_{i=1}^{n} ((x_i-x_0)^2 + (y_i-y_0)^2 - R^2)^2 \\
& = (50 - x_0)^2 + (1.3 - y_0)^2 - R^2 \quad (-100 + 2x_0) \\
& + 2((150 - x_0)^2 + (1 - y_0)^2 - R^2) \quad (-100 + 2x_0) \\
& + 2((250 - x_0)^2 + (22.5 - y_0)^2 - R^2) \quad (-500 + 2x_0) = 0
\end{align*}
\]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).

\[
\begin{align*}
eq & : = \text{diff}(F(x_0, y_0, R), x_0) = 0; \\
eq 1 & : = 4((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) \quad (-100 + 2x_0) \\
& + 2((150 - x_0)^2 + (1 - y_0)^2 - R^2) \quad (-100 + 2x_0) \\
& + 2((250 - x_0)^2 + (22.5 - y_0)^2 - R^2) \quad (-500 + 2x_0) = 0 \\
eq & : = \text{diff}(F(x_0, y_0, R), y_0) = 0; \\
eq 2 & : = 4((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) \quad (-100 + 2x_0) \\
& + 2((150 - x_0)^2 + (1 - y_0)^2 - R^2) \quad (-100 + 2x_0) \\
& + 2((250 - x_0)^2 + (22.5 - y_0)^2 - R^2) \quad (-500 + 2x_0) = 0 \\
eq & : = \text{diff}(F(x_0, y_0, R), R) = 0; \\
eq 3 & : = -4((50 - x_0)^2 + (1.3 - y_0)^2 - R^2) \quad R - 4((150 - x_0)^2 + (1 - y_0)^2 - R^2) \quad R \\
& - 4((250 - x_0)^2 + (22.5 - y_0)^2 - R^2) \quad R = 0
\end{align*}
\]

\[
\text{sol} := \text{solve}([\text{eq1}, \text{eq2}, \text{eq3}], [x_0, y_0, R]);
\]

\[
\begin{align*}
\text{sol} & := \{R = 0, x_0 = 150.0010835 - 141.5205037 I, y_0 = 6.450473664 - 15.06751095 I \}, \\
& \{R = 0, x_0 = 150.0010835 + 141.5205037 I, y_0 = 6.450473664 + 15.06751095 I \}, \\
& \{R = 0, y_0 = 11.83753064, x_0 = 150.0043895 \}, \\
& \{R = 0, x_0 = 150.7612906 - 8.732759880 I, y_0 = 6.562442597 + 82.18099726 I \}, \\
& \{R = 0, x_0 = 150.7612906 + 8.732759880 I, y_0 = 6.562442597 - 82.18099726 I \}, \\
& \{y_0 = 470.3197248, x_0 = 101.4075092, R = -471.8286068 \}.
\end{align*}
\]
\{(y0 = 470.3197248, x0 = 101.4075092, R = 471.8286068)\}

> \texttt{circ := (x-x0)^2 + (y-y0)^2 = R^2}

> \texttt{with(plots)};

\{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported, polyhedr plot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, sparsematplot, sphereplot, surfdata, textplot, textplot3d, tubeplot\}

> \texttt{p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000)};

> \texttt{with(stats[statplots])};

\{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift, yzexchange, zsframe, zshift\}

> \texttt{p2 := scatterplot(xx, yy, color=black)};

> \texttt{display([p1, p2], scaling=constrained)};
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx, and the list of y coordinates as yy. The number of points in the list is n.
The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order. There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "Futlock VII, B5E".

> restart;
> xx := [3.30, 60, 90, 120, 150, 180, 210, 240, 270, 300];
> yy := [6.2, 2.6, 8, 6.0, 1, 4.8, 10.6, 18.0, 28.2, 40.6];
> cir := (x, y) → ((x-x0)^2+(y-y0)^2)=R^2;
> n := 11;
> for i from 1 to n-2 do
> eq1 := cir(xx[i],yy[i]);
> eq2 := cir(xx[i+1],yy[i+1]);
> eq3 := cir(xx[i+2],yy[i+2]);
> print(i, solve({eq1, eq2, eq3}, {x0, y0, R}));
> od:

1. \( y0 = 395.2454545 \) \( x0 = 68.61272727 \) \( R = -394.5394729 \)
2. \( y0 = 564.3525000 \) \( x0 = 78.75750000 \) \( R = -563.8370948 \)
3. \( y0 = -2249.900000 \) \( x0 = 59.99600000 \) \( R = -2250.700000 \)
4. \( y0 = -2249.900000 \) \( x0 = 59.99600000 \) \( R = 2250.700000 \)
5. \( y0 = 116.2525000 \) \( x0 = 562.9250000 \) \( R = -562.9374738 \)
6. \( y0 = 116.2525000 \) \( x0 = 562.9250000 \) \( R = 562.9374738 \)
7. \( y0 = 325.1857143 \) \( x0 = 124.1771429 \) \( R = -325.2125417 \)
8. \( y0 = 466.8200000 \) \( x0 = 106.2368000 \) \( R = -467.8712324 \)
9. \( y0 = 466.8200000 \) \( x0 = 106.2368000 \) \( R = 467.8712324 \)
10. \( y0 = 117.8761111 \) \( x0 = 406.5907170 \) \( y0 = 406.6166667 \)
11. \( y0 = 117.8761111 \) \( x0 = 406.5907170 \) \( y0 = 406.6166667 \)
12. \( y0 = 833.6333333 \) \( x0 = 1.15822222 \) \( y0 = -849.1184560 \)
13. \( R = 849.1164560 \) \( y0 = 833.6333333 \) \( x0 = 1.15822222 \)
14. \( y0 = 146.8832889 \) \( y0 = 368.5533333 \) \( R = -361.9366189 \)

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Port side
Floor VI
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The data I entered was labelled, 

\[
\begin{align*}
xx & := [0, 10, 20, 30, 40, 43, 50, 60, 70, 80, 85, 90, 100, 110, 120, 130, 138]; \\
yy & := [1.1, 0.8, 0.3, 0, 0.2, 0.8, 1.3, 2.4, 4.3, 5.9, 8.5, 11.4, 11.8, 14.5, 18.5, 23.5]; \\
n & := 17;
\end{align*}
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]
eq3 := 2 (x^2 + (1.1 - y)^2 - R^2) (-2.2 + 2 y) + 2 ((10 - x)^2 + (8 - y)^2 - R^2) (-1.6 + 2 y) 
+ 2 ((20 - x)^2 + (3 - y)^2 - R^2) (-6.2 + 2 y) + 4 ((30 - x)^2 + y^2 - R^2) y 
+ 4 ((40 - x)^2 + y^2 - R^2) y + 2 ((43 - x)^2 + (2 - y)^2 - R^2) (-4.2 + 2 y) 
+ 2 ((50 - x)^2 + (8 - y)^2 - R^2) (-1.6 + 2 y) 
+ 2 ((60 - x)^2 + (1.3 - y)^2 - R^2) (-2.6 + 2 y) 
+ 2 ((70 - x)^2 + (2.4 - y)^2 - R^2) (-4.8 + 2 y) 
+ 2 ((80 - x)^2 + (4.3 - y)^2 - R^2) (-8.6 + 2 y) 
+ 2 ((85 - x)^2 + (5.9 - y)^2 - R^2) (-11.8 + 2 y) 
+ 2 ((90 - x)^2 + (8.5 - y)^2 - R^2) (-17.0 + 2 y) 
+ 2 ((100 - x)^2 + (11.4 - y)^2 - R^2) (-22.8 + 2 y) 
+ 2 ((110 - x)^2 + (11.8 - y)^2 - R^2) (-23.6 + 2 y) 
+ 2 ((120 - x)^2 + (14.5 - y)^2 - R^2) (-29.0 + 2 y) 
+ 2 ((130 - x)^2 + (18.5 - y)^2 - R^2) (-37.0 + 2 y) 
+ 2 ((138 - x)^2 + (23.5 - y)^2 - R^2) (-47.0 + 2 y) = 0

> eq3 := diff(R(x0, y0, R), R) = 0;

> sol := solve({eq1, eq2, eq3}, {x0, y0, R});
with plots;

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, 
  contourplot, contourplot3d, coordinate, coordplot3d, cylinderplot, densityplot, display, display3d, 
  fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot, 
  listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto, 
  pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra, supported, 
  polyhedraptor, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve, 
  sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

> pl := implicitplot(subs(sol[7], circ), x = 0 .. 400, y = -300 .. 300, numpoints = 4000);

> with(stats[statplots]);

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift, 
  yzexchange, zscale, zshift]

> p2 := scatterplot(xx, yy, color = black);

> display([p1, p2], scaling = constrained);
B6E
Port side
Futlock VI
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled.

\[
\text{restart;}
\]
\[
xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280];
\]
\[
xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280];
\]
\[
yy := [0, 0.1, 0.7, 0.7, 0.2, 0, 1, 2, 3, 3.3, 4.8, 7.2, 10, 12.6, 15.8, 19, 22.1, 25.5, 29.7, 33.3, 37.4, 42.2, 46.7, 51.9, 57.3, 62.8, 68.2, 74.2, 80.6, 90.6];
\]
\[
yy := [0, 0.1, 0.7, 0.7, 0.2, 0, 1, 2, 3, 3.3, 4.8, 7.2, 10, 12.6, 15.8, 19, 22.1, 25.5, 29.7, 33.3, 37.4, 42.2, 46.7, 51.9, 57.3, 62.8, 68.2, 74.2, 80.6, 90.6];
\]
\[
n := 29;
\]

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x-x_0)^2 + (y-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x0, y0, R) \rightarrow \sum_{i=1}^{n} ((xx[i]-x0)^2 + (yy[i]-y0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, \) and \(R\).

\[
eq 0;
\]

\[
eq 2 ((260 - x0)^2 + (74.2 - y0)^2 - R^2)^2 (-520 + 2 x0)
\]
\[
+ 2 ((270 - x0)^2 + (80.6 - y0)^2 - R^2)^2 (-540 + 2 x0)
\]
\[
+ 2 ((120 - x0)^2 + (90.6 - y0)^2 - R^2)^2 (-560 + 2 x0)
\]
\[
+ 2 ((20 - x0)^2 + (7 - y0)^2 - R^2)^2 (-60 + 2 x0) + 2 ((30 - x0)^2 + (7 - y0)^2 - R^2)^2 (-60 + 2 x0)
\]
\[
+ 2 ((40 - x0)^2 + (2 - y0)^2 - R^2)^2 (-80 + 2 x0) + 2 ((50 - x0)^2 + y0^2 - R^2)^2 (-100 + 2 x0)
\]
\[
+ 2 ((60 - x0)^2 + (1 - y0)^2 - R^2)^2 (-120 + 2 x0)
\]
\[
+ 2 ((70 - x0)^2 + (2.1 - y0)^2 - R^2)^2 (-140 + 2 x0)
\]
\[
+ 2 ((80 - x0)^2 + (3.3 - y0)^2 - R^2)^2 (-160 + 2 x0)
\]
\[
+ 2 ((90 - x0)^2 + (4.8 - y0)^2 - R^2)^2 (-180 + 2 x0)
\]
\[
+ 2 ((100 - x0)^2 + (7.2 - y0)^2 - R^2)^2 (-200 + 2 x0)
\]
\[
+ 2 ((110 - x0)^2 + (10 - y0)^2 - R^2)^2 (-220 + 2 x0)
\]
\[
+ 2 ((120 - x0)^2 + (12.6 - y0)^2 - R^2)^2 (-240 + 2 x0)
\]

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\[ + 2 \left( 130 - x^0 \right)^2 + \left( 15.8 - y^0 \right)^2 - R^2 \right) \left( -260 + 2 x^0 \right) \\
+ 2 \left( 140 - x^0 \right)^2 + \left( 19 - y^0 \right)^2 - R^2 \right) \left( -280 + 2 x^0 \right) \\
+ 2 \left( 150 - x^0 \right)^2 + \left( 22.1 - y^0 \right)^2 - R^2 \right) \left( -300 + 2 x^0 \right) \\
+ 2 \left( 160 - x^0 \right)^2 + \left( 25.5 - y^0 \right)^2 - R^2 \right) \left( -320 + 2 x^0 \right) \\
+ 2 \left( 170 - x^0 \right)^2 + \left( 29.7 - y^0 \right)^2 - R^2 \right) \left( -340 + 2 x^0 \right) \\
+ 2 \left( 180 - x^0 \right)^2 + \left( 33.3 - y^0 \right)^2 - R^2 \right) \left( -360 + 2 x^0 \right) \\
+ 2 \left( 190 - x^0 \right)^2 + \left( 37.4 - y^0 \right)^2 - R^2 \right) \left( -380 + 2 x^0 \right) \\
+ 2 \left( 200 - x^0 \right)^2 + \left( 42.2 - y^0 \right)^2 - R^2 \right) \left( -400 + 2 x^0 \right) \\
+ 2 \left( 210 - x^0 \right)^2 + \left( 46.7 - y^0 \right)^2 - R^2 \right) \left( -420 + 2 x^0 \right) \\
+ 2 \left( 220 - x^0 \right)^2 + \left( 51.9 - y^0 \right)^2 - R^2 \right) \left( -440 + 2 x^0 \right) \\
+ 2 \left( 230 - x^0 \right)^2 + \left( 57.3 - y^0 \right)^2 - R^2 \right) \left( -460 + 2 x^0 \right) \\
+ 2 \left( 240 - x^0 \right)^2 + \left( 62.8 - y^0 \right)^2 - R^2 \right) \left( -480 + 2 x^0 \right) \\
+ 2 \left( 250 - x^0 \right)^2 + \left( 68.2 - y^0 \right)^2 - R^2 \right) \left( -500 + 2 x^0 \right) \\
+ 2 \left( \left( 10 - x^0 \right)^2 + \left( 1.1 - y^0 \right)^2 - R^2 \right) \left( -20 + 2 x^0 \right) + 4 \left( x^0^2 + y^0^2 - R^2 \right) x^0 = 0 \\
\]

\[ eq2 := \text{diff}(F(x^0, y^0, R), y^0) = 0; \]

\[ eq2 := 2 \left( 10 - x^0 \right)^2 + \left( 1.1 - y^0 \right)^2 - R^2 \right) \left( -2 + 2 x^0 \right) + 4 \left( x^0^2 + y^0^2 - R^2 \right) y^0 \\
+ 2 \left( 60 - x^0 \right)^2 + \left( 1.1 - y^0 \right)^2 - R^2 \right) \left( -2 + 2 x^0 \right) + 2 \left( 70 - x^0 \right)^2 + \left( 2.1 - y^0 \right)^2 - R^2 \right) \left( -4.2 + 2 x^0 \right) \\
+ 2 \left( 80 - x^0 \right)^2 + \left( 3.3 - y^0 \right)^2 - R^2 \right) \left( -6.6 + 2 y^0 \right) \\
+ 2 \left( 90 - x^0 \right)^2 + \left( 4.8 - y^0 \right)^2 - R^2 \right) \left( -9.6 + 2 y^0 \right) \\
+ 2 \left( 100 - x^0 \right)^2 + \left( 7.2 - y^0 \right)^2 - R^2 \right) \left( -14.4 + 2 y^0 \right) \\
+ 2 \left( 110 - x^0 \right)^2 + \left( 10 - y^0 \right)^2 - R^2 \right) \left( -20 + 2 y^0 \right) \\
- 2 \left( 120 - x^0 \right)^2 + \left( 12.6 - y^0 \right)^2 - R^2 \right) \left( -25.2 + 2 y^0 \right) \\
+ 2 \left( 130 - x^0 \right)^2 + \left( 15.8 - y^0 \right)^2 - R^2 \right) \left( -31.6 + 2 y^0 \right) \\
+ 2 \left( 140 - x^0 \right)^2 + \left( 19 - y^0 \right)^2 - R^2 \right) \left( -38 + 2 y^0 \right) \\
+ 2 \left( 150 - x^0 \right)^2 + \left( 22.1 - y^0 \right)^2 - R^2 \right) \left( -44.2 + 2 y^0 \right) \\
+ 2 \left( 160 - x^0 \right)^2 + \left( 25.5 - y^0 \right)^2 - R^2 \right) \left( -51.0 + 2 y^0 \right) \\
+ 2 \left( 170 - x^0 \right)^2 + \left( 29.7 - y^0 \right)^2 - R^2 \right) \left( -59.4 + 2 y^0 \right) \\
+ 2 \left( 180 - x^0 \right)^2 + \left( 33.3 - y^0 \right)^2 - R^2 \right) \left( -66.6 + 2 y^0 \right) \\
+ 2 \left( 190 - x^0 \right)^2 + \left( 37.4 - y^0 \right)^2 - R^2 \right) \left( -74.8 + 2 y^0 \right) \\
+ 2 \left( 200 - x^0 \right)^2 + \left( 42.2 - y^0 \right)^2 - R^2 \right) \left( -84.4 + 2 y^0 \right) \\
+ 2 \left( 210 - x^0 \right)^2 + \left( 46.7 - y^0 \right)^2 - R^2 \right) \left( -93.4 + 2 y^0 \right) \\
+ 2 \left( 220 - x^0 \right)^2 + \left( 51.9 - y^0 \right)^2 - R^2 \right) \left( -103.8 + 2 y^0 \right) \\
- 2 \left( 230 - x^0 \right)^2 + \left( 57.3 - y^0 \right)^2 - R^2 \right) \left( -114.6 + 2 y^0 \right) \\
\]
\[
+ 2 ((240 - x0)^2 + (62.8 - y0)^2 - R^2) (-125.6 + 2 y0) \\
+ 2 ((250 - x0)^2 + (68.2 - y0)^2 - R^2) (-136.4 + 2 y0) \\
+ 2 ((260 - x0)^2 + (74.2 - y0)^2 - R^2) (-148.4 + 2 y0) \\
+ 2 ((270 - x0)^2 + (80.6 - y0)^2 - R^2) (-161.2 + 2 y0) \\
+ 2 ((280 - x0)^2 + (90.6 - y0)^2 - R^2) (-181.2 + 2 y0) \\
+ 2 ((290 - x0)^2 + (.7 - y0)^2 - R^2) (-1.4 + 2 y0) \\
+ 2 ((300 - x0)^2 + (.7 - y0)^2 - R^2) (-1.4 + 2 y0) + 2 ((40 - x0)^2 + (.2 - y0)^2 - R^2) (-4 + 2 y0) \]
\[
+ 4 ((50 - x0)^2 + y0^2 - R^2) y0 = 0
\]

eq3 := d.\text{iff}(R(x0, y0, R), R) = 0;

\[
eq q 3 := -4 \begin{array}{c}
((20 - x0)^2 + (.7 - y0)^2 - R^2) R \\
-4 ((10 - x0)^2 + (.1 - y0)^2 - R^2) R - 4 (x0^2 + y0^2 - R^2) R - 4 ((50 - x0)^2 + y0^2 - R^2) R \\
-4 ((60 - x0)^2 + (1 - y0)^2 - R^2) R - 4 ((70 - x0)^2 + (2.1 - y0)^2 - R^2) R \\
-4 ((80 - x0)^2 + (3.3 - y0)^2 - R^2) R - 4 ((90 - x0)^2 + (4.8 - y0)^2 - R^2) R \\
-4 ((100 - x0)^2 + (7.2 - y0)^2 - R^2) R - 4 ((110 - x0)^2 + (10 - y0)^2 - R^2) R \\
-4 ((120 - x0)^2 + (12.6 - y0)^2 - R^2) R - 4 ((130 - x0)^2 + (15.8 - y0)^2 - R^2) R \\
-4 ((140 - x0)^2 + (19 - y0)^2 - R^2) R - 4 ((150 - x0)^2 + (22.1 - y0)^2 - R^2) R \\
-4 ((160 - x0)^2 + (25.5 - y0)^2 - R^2) R - 4 ((170 - x0)^2 + (29.7 - y0)^2 - R^2) R \\
-4 ((180 - x0)^2 + (33.3 - y0)^2 - R^2) R - 4 ((190 - x0)^2 + (37.4 - y0)^2 - R^2) R \\
-4 ((200 - x0)^2 + (42.2 - y0)^2 - R^2) R - 4 ((210 - x0)^2 + (46.7 - y0)^2 - R^2) R \\
-4 ((220 - x0)^2 + (51.9 - y0)^2 - R^2) R - 4 ((230 - x0)^2 + (57.3 - y0)^2 - R^2) R \\
-4 ((240 - x0)^2 + (62.8 - y0)^2 - R^2) R - 4 ((250 - x0)^2 + (68.2 - y0)^2 - R^2) R \\
-4 ((260 - x0)^2 + (74.2 - y0)^2 - R^2) R - 4 ((270 - x0)^2 + (80.6 - y0)^2 - R^2) R \\
-4 ((280 - x0)^2 + (90.6 - y0)^2 - R^2) R - 4 ((40 - x0)^2 + (.2 - y0)^2 - R^2) R = 0
\end{array}
\]

> sol := solve([eq1, eq2, eq3], {x0, y0, R});

\[
sol := \{R = 0, y0 = 23.82545205 - 46.19027222 I, x0 = 140.0248291 - 145.0808668 I\},
\{R = 0, x0 = 140.0248291 + 145.0808668 I, y0 = 23.82545205 + 46.19027222 I\},
\{R = 0, y0 = 34.84688297, x0 = 140.0336752\},
\{R = 0, y0 = 25.31866690 + 84.95860984 I, x0 = 144.2792393 - 26.85608704 I\},
\{R = 0, y0 = 25.31866690 - 84.95860984 I, x0 = 144.2792393 + 26.85608704 I\},
\{y0 = 438.6594413, x0 = 12.89130673, R = -439.2062593\},
\{R = 439.2062593, y0 = 438.6594413, x0 = 12.89130673\}
\]

> circ := (x-x0)^2 + (y-y0)^2 = R^2

F; with(plots);

[animate, animate3d, animatecurve, changecolors, zoomplot, complexplot, complexplot3d, conformal,
contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematplot, sphereplot, surfdata, textplot, textplot3d, tubeplot

> p1:=implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000):

> with(stats[statplots]):

| boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift,
| yzexchange, zscale, zshift |

> p2:=scatterplot(xx, yy, color=black):

> display([p1, p2], scaling=constrained);
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a
list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The
data I entered was labelled.

> restart;
> xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270];
> yy := [0, 7.1, 1.4, 8.1, 12.6, 22.1, 33.3, 46.7, 62.8, 80.6];
> n := 10;

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\)
i = 1 ... n, then the n quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to
hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these
quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the
resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\).

> eq1 := diff(F(x0, y0, R), x0) = 0;

\[
eq 1 := 4 (x_0^2 + y_0^2 - R^2) x_0 + 2 ((30 - x_0)^2 + (.7 - y_0)^2 - R^2) (-60 + 2 x_0)
\]

\[
+ 2 ((60 - x_0)^2 + (1 - y_0)^2 - R^2) (-120 + 2 x_0)
\]

\[
+ 2 ((90 - x_0)^2 + (4.8 - y_0)^2 - R^2) (-180 + 2 x_0)
\]

\[
+ 2 ((120 - x_0)^2 + (12.6 - y_0)^2 - R^2) (-240 + 2 x_0)
\]

\[
+ 2 ((150 - x_0)^2 + (22.1 - y_0)^2 - R^2) (-300 + 2 x_0)
\]

\[
+ 2 ((180 - x_0)^2 + (33.3 - y_0)^2 - R^2) (-360 + 2 x_0)
\]

\[
+ 2 ((210 - x_0)^2 + (46.7 - y_0)^2 - R^2) (-420 + 2 x_0)
\]

\[
+ 2 ((240 - x_0)^2 + (62.8 - y_0)^2 - R^2) (-480 + 2 x_0)
\]

\[
+ 2 ((270 - x_0)^2 + (80.6 - y_0)^2 - R^2) (-540 + 2 x_0) = 0
\]

> eq2 := diff(F(x0, y0, R), y0) = 0;

\[
eq 2 := 4 (x_0^2 + y_0^2 - R^2) y_0 + 2 ((30 - x_0)^2 + (.7 - y_0)^2 - R^2) (-1.4 + 2 y_0)
\]

\[
+ 2 ((60 - x_0)^2 + (1 - y_0)^2 - R^2) (-2 + 2 y_0) + 2 ((90 - x_0)^2 + (4.8 - y_0)^2 - R^2) (-9.6 + 2 y_0)
\]

\[
+ 2 ((120 - x_0)^2 + (12.6 - y_0)^2 - R^2) (-25.2 + 2 y_0)
\]

\[
+ 2 ((150 - x_0)^2 + (22.1 - y_0)^2 - R^2) (-44.2 + 2 y_0)
\]

\[
+ 2 ((180 - x_0)^2 + (33.3 - y_0)^2 - R^2) (-66.6 + 2 y_0)
\]

\[
+ 2 ((210 - x_0)^2 + (46.7 - y_0)^2 - R^2) (-93.4 + 2 y_0)
\]
\[ + 2 \left( (240 - x^2)^2 + (62.8 - y^2)^2 - R^2 \right) (-125.6 + 2 y^2) + 2 \left( (270 - x^2)^2 + (80.6 - y^2)^2 - R^2 \right) (-161.2 + 2 y^2) = 0 \]

\[ \text{eq3} := \text{diff}(F(x0, y0, R), R) = 0; \]

\[ \text{eq3} := -4 \left( x^2 + y^2 - R^2 \right) R - 4 \left( (30 - x^2)^2 + (7 - y^2)^2 - R^2 \right) R \]
\[ - 4 \left( (60 - x^2)^2 + (1 - y^2)^2 - R^2 \right) R - 4 \left( (90 - x^2)^2 + (4.8 - y^2)^2 - R^2 \right) R \]
\[ - 4 \left( (120 - x^2)^2 + (12.6 - y^2)^2 - R^2 \right) R - 4 \left( (150 - x^2)^2 + (22.1 - y^2)^2 - R^2 \right) R \]
\[ - 4 \left( (180 - x^2)^2 + (33.3 - y^2)^2 - R^2 \right) R - 4 \left( (210 - x^2)^2 + (46.7 - y^2)^2 - R^2 \right) R \]
\[ - 4 \left( (240 - x^2)^2 + (62.8 - y^2)^2 - R^2 \right) R - 4 \left( (270 - x^2)^2 + (80.6 - y^2)^2 - R^2 \right) R = 0 \]

\[ \text{sol} := \text{solve}\{\text{eq1}, \text{eq2}, \text{eq3}\}, \{x0, y0, R\}; \]

\[ \text{sol} := \{ R = 0, y0 = 22.68544861 - 45.27184345 I, x0 = 135.0260953 + 149.4260746 I \}, \]
\[ \{ R = 0, y0 = 22.68544861 + 45.27184345 I, x0 = 135.0260953 + 149.4260746 I \}, \]
\[ \{ R = 0, x0 = 135.0275852, y0 = 33.74666827 \}, \]
\[ \{ R = 0, y0 = 24.06361203 + 87.48138064 I, x0 = 139.1162733 - 26.32383892 I \}, \]
\[ \{ R = 0, y0 = 24.06361203 - 87.48138064 I, x0 = 139.1162733 + 26.32383892 I \}, \]
\[ \{ x0 = 12.32847499, y0 = 443.1805870, R = -443.6996586 \}, \]

\[ \{ x0 = 12.32847499, y0 = 443.1805870, R = 443.6996586 \} \]

\[ \text{circ} := (x - x0)^2 + (y - y0)^2 = R^2 \]

\[ \text{with(plots);} \]

\[ \text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,} \]
\[ \text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,} \]
\[ \text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,} \]
\[ \text{listcontplot3d, listdensityplot, listplot, listsplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,} \]
\[ \text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported,} \]
\[ \text{polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,} \]
\[ \text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot} \]

\[ \text{pl1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);} \]

\[ \text{with(plots[statplots]);} \]

\[ \text{boxplot, histogram, scatterplot, xscale, xshift, yexchange, xexchange, yscale, yshift,} \]
\[ \text{yexchange, xscale, xshift} \]

\[ \text{pl2 := scatterplot(xx, yy, color=black);} \]

\[ \text{display([pl1, pl2], scaling=constrained);} \]
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, \( \texttt{restart;}, \)
\( \texttt{xx := [50, 100, 150, 200, 250]}; \)
\( \texttt{yy := [0, 7.2, 22.1, 42.2, 68.2]} \);
\( \texttt{n := 5}; \)

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \([x_i, y_i]\), \(i = 1 .. n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[ F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2 \]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

\[ \texttt{eq1 := diff(F(x0, y0, R), x0)} = 0; \]
\[ eq1 := 2 ((50 - x0)^2 + y0^2 - R^2)(-100 + 2 x0) \]
\[ + 2 ((100 - x0)^2 + (7.2 - y0)^2 - R^2)(-200 + 2 x0) \]
\[ + 2 ((150 - x0)^2 + (22.1 - y0)^2 - R^2)(-300 + 2 x0) \]
\[ + 2 ((200 - x0)^2 + (42.2 - y0)^2 - R^2)(-400 + 2 x0) \]
\[ + 2 ((250 - x0)^2 + (68.2 - y0)^2 - R^2)(-500 + 2 x0) = 0 \]

\[ \texttt{eq2 := diff(F(x0, y0, R), y0)} = 0; \]
\[ eq2 := 4 ((50 - x0)^2 + y0^2 - R^2) y0 + 2 ((100 - x0)^2 + (7.2 - y0)^2 - R^2)(-14.4 + 2 y0) \]
\[ + 2 ((150 - x0)^2 + (22.1 - y0)^2 - R^2)(-44.2 + 2 y0) \]
\[ + 2 ((200 - x0)^2 + (42.2 - y0)^2 - R^2)(-84.4 + 2 y0) \]
\[ + 2 ((250 - x0)^2 + (68.2 - y0)^2 - R^2)(-136.4 + 2 y0) = 0 \]

\[ \texttt{eq3 := diff(F(x0, y0, R), R)} = 0; \]
\[ eq3 := -4 ((50 - x0)^2 + y0^2 - R^2) R - 4 ((100 - x0)^2 + (7.2 - y0)^2 - R^2) R \]
\[ - 4 ((150 - x0)^2 + (22.1 - y0)^2 - R^2) R - 4 ((200 - x0)^2 + (42.2 - y0)^2 - R^2) R \]
\[ - 4 ((250 - x0)^2 + (68.2 - y0)^2 - R^2) R = 0 \]

\[ \texttt{sol := solve([eq1, eq2, eq3], [x0, y0, R]);} \]
\[ sol := [R = 0, y0 = 25.80254439 - 42.23904326 I, x0 = 150.0084262 - 122.5392474 I]. \]
\[ R = 0, x0 = 150.0084262 + 122.5392474 I, y0 = 25.80254439 + 42.23904326 I, \]
\[ R = 0, y0 = 32.14099992, x0 = 150.0086634, \]
\[ R = 0, x0 = 152.6113312 - 24.48304479 I, y0 = 26.73964964 + 71.25671270 I, \]
\[ R = 0, y0 = 26.73964964 - 71.25671270 I, x0 = 152.6113312 + 24.48304479 I, \]
\[ x0 = -4.273868264, y0 = 482.2209693, R = 485.5774376, \]
\[ x0 = -4.273868264, y0 = 482.2209693, R = 485.5774376 \]
\[ \texttt{circ} := (x-x0)^2 + (y-y0)^2 = R^2; \]
\[ \texttt{circ} := (x-x0)^2 + (y-y0)^2 = R^2 \]
\[ \texttt{with(plots);} \]
\[ \text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,} \]
\[ \text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,} \]
\[ \text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,} \]
\[ \text{listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,} \]
\[ \text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra-supported,} \]
\[ \text{polyhedraplot, replot, rootcurve, semilogplot, setoptions, setoptions3d, spacecurve,} \]
\[ \text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot} \]
\[ \texttt{p1 := implicitplot(subs(sol[7], circ), x = 0..400, y = -300..300, numpoints = 4000);} \]
\[ \texttt{w} \]
\[ \texttt{with(stats[statplots]);} \]
\[ \text{boxplot, histogram, scatterplot, xscale, xshift, yexchange, zexchange, yscale, yshift,} \]
\[ \text{yexchange, xscale, zshift} \]
\[ \texttt{p2 := scatterplot(xx, yy, color = black);} \]
\[ \texttt{display([p1, p2], scaling = constrained);} \]
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, \( \text{????????} \).

\[ \text{restart; } \]
\[ xx := [50, 150, 250] ; \]
\[ yy := [0, 22.1, 68.2] ; \]
\[ n := 3 ; \]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \( [x_i, y_i] \), \( i = 1 \ldots n \), then the \( n \) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \( x_0, y_0, \) and \( R \) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[ F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2 \]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).

\[ \text{sol := solve (solve (solve (eq1, eq2, eq3), (x0, y0, R)), (x0, y0, R)) ;} \]

\[ \text{sol := (R = 0, y0 = 28.10498260 + 48.48138477 I, x0 = 150.0027001 - 141.4873754 I}, \]
\[ \{ R = 0, x0 = 150.0027001 + 141.4873754 I, y0 = 28.10498260 + 48.48138477 I \}, \]
\[ \{ R = 0, x0 = 150.0154459, y0 = 34.0292614 I } \}, \]
\[ \{ R = 0, y0 = 28.97303723 + 82.19169367 I, x0 = 152.4395323 - 28.10369893 I \}, \]
\[ \{ R = 0, y0 = 28.97303723 - 82.19169367 I, x0 = 152.4395323 + 28.10369893 I \}, \]
\[ \{ x0 = -6.555892517, y0 = 493.2170833, R = -496.4493966 \}, \]
\[ \{ x0 = -6.555892517, y0 = 493.2170833, R = 496.4493966 \} \]
\[ \text{circ} := (x-x0)^2 + (y-y0)^2 = R^2; \]

\[
\text{circ} := (x - x0)^2 + (y - y0)^2 = R^2
\]

\[
> \text{with(plots)};
\]

\[
\text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,}
\]

\[\text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,}
\]

\[\text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequl, listcontourplot,}
\]

\[\text{listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,}
\]

\[\text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported,}
\]

\[\text{polyhedralplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,}
\]

\[\text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot}\]

\[> \text{p1 := implicitplot(subs(aol[7], circ), x=0..400, y=-300..300, numpoints=}
\]

\[\text{4000);}
\]

\[> \text{with(stats[statplots]);}
\]

\[\text{[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,}
\]

\[\text{yexchange, zscale, zshift]}\]

\[> \text{p2 := scatterplot(xx, yy, color=black);}
\]

\[> \text{display([p1, p2], scaling=constrained});}
\]
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx, and the list of y coordinates as yy. The number of points in the list is n. The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "Futrock VI, B6E".

> restart;
> xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280];
> yy := [0, 0.1, 0.7, 0.7, 0.2, 0.1, 2.1, 3.3, 4.8, 7.2, 10, 12.6, 15.8, 19, 22.1, 25.5, 29.7, 33.3, 37.4, 42.2, 46.7, 51.9, 57.3, 62.8, 68.2, 74.2, 80.6, 90.6];
> cir := (x, y) \rightarrow ((x-x0)^2 + (y-y0)^2) = R^2;
> n := 29;
> for i from 1 to n-2 do
> eq1 := cir(xx[i], yy[i]);
> eq2 := cir(xx[i+1], yy[i+1]);
> eq3 := cir(xx[i+2], yy[i+2]);
> print(i, solve([eq1, eq2, eq3], [x0, y0, R]));
> end;

1. \{ y0 = 200.4700000, x0 = 2.995800000, R = -200.4923832 \},
   \{ y0 = 200.4700000, x0 = 2.995800000, R = 200.4923832 \}
2. \{ y0 = -166.2666667, R = -167.0415151, x0 = 25, \},
   \{ y0 = -166.2666667, R = 167.0415151, x0 = 25, \}
3. \{ y0 = -199.5500000, R = -200.3124122, x0 = 25, \},
   \{ R = 200.3124122, y0 = -199.5500000, x0 = 25 \}
4. \{ y0 = 334.0166667, x0 = 51.67833333, R = -334.0208832 \},
   \{ y0 = 334.0166667, x0 = 51.67833333, R = 334.0208832 \}
5. \{ x0 = 46.67333333, y0 = 83.76666667, R = -83.83269741 \},
   \{ x0 = 46.67333333, y0 = 83.76666667, R = 83.83269741 \}
6. \{ y0 = 1012.050000, x0 = -46.15500000, R = -1016.607587 \},
   \{ y0 = 1012.050000, x0 = -46.15500000, R = 1016.607587 \}
7. \{ R = -1019.927452, x0 = -46.51800000, y0 = 1015.350000 \}
{ R = 1019.927452, x0 = -46.51800000, y0 = 1015.350000 } 
8. { R = -342.5601987, x0 = 34.19000000, y0 = 342.7833333 }, 
{ R = 342.5601987, x0 = 34.19000000, y0 = 342.7833333 } 
9. { y0 = 120.3611111, x0 = 67.55333333, R = -117.7209550 }, 
{ y0 = 120.3611111, x0 = 67.55333333, R = 117.7209550 } 
10. { R = -275.8639509, y0 = 274.2000000, x0 = 30.63200000 }, 
{ R = 275.8639509, y0 = 274.2000000, x0 = 30.63200000 } 
11. { y0 = -526.5000000, x0 = 254.8280000, R = -555.7044175 }, 
{ y0 = -526.5000000, x0 = 254.8280000, R = 555.7044175 } 
12. { R = -188.2596759, x0 = 67.64533333, y0 = 193.4333333 }, 
{ R = 188.2596759, x0 = 67.64533333, y0 = 193.4333333 } 

13 
14. { x0 = 486.2480000, y0 = -1080.250000, R = -1152.492187 }, 
{ x0 = 486.2480000, y0 = -1080.250000, R = 1152.492187 } 
15. { x0 = 30.24833333, y0 = 390.7166667, R = -387.5805834 }, 
{ x0 = 30.24833333, y0 = 390.7166667, R = 387.5805834 } 
16. { x0 = 105.7170000, y0 = 168.7500000, R = -153.1900995 }, 
{ x0 = 105.7170000, y0 = 168.7500000, R = 153.1900995 } 
17. { y0 = -162.4666667, x0 = 244.8280000, R = -206.2213795 }, 
{ R = 206.2213795, y0 = -162.4666667, x0 = 244.8280000 } 
18. { x0 = 91.63480000, y0 = 263.0700000, R = -246.1760782 }, 
{ x0 = 91.63480000, y0 = 263.0700000, R = 246.1760782 } 
19. { x0 = 113.917143, y0 = 208.7214286, R = -187.4554509 }, 
{ x0 = 113.917143, y0 = 208.7214286, R = 187.4554509 } 
20. { y0 = -363.2833333, x0 = 388.4800000, R = -447.1481231 }, 
{ x0 = -363.2833333, x0 = 388.4800000, R = 447.1481231 } 
21. { x0 = 124.5014286, y0 = 223.3357143, R = -196.2401113 }, 
{ x0 = 124.5014286, y0 = 223.3357143, R = 196.2401113 } 
22. { y0 = 692.4000000, x0 = -119.4120000, R = -724.8729239 }, 
{ R = 724.8729239, y0 = 692.4000000, x0 = -119.4120000 } 
23. { R = -1477.158363, x0 = 1354.350000, x0 = -476.8650000 }, 
{ R = 1477.158363, y0 = 1354.350000, x0 = -476.8650000 } 
24. { R = -1477.158363, y0 = -1234.250000, x0 = 946.8650000 }, 
{ R = 1477.158363, y0 = -1234.250000, x0 = 946.8650000 } 
25. { y0 = 289.1666667, x0 = 124.2200000, R = -254.2574997 }, 
{ y0 = 289.1666667, x0 = 124.2200000, R = 254.2574997 }
26. \( \{ y_0 = 420.4000000, R = -407.2754233, x_0 = 45.48000000 \} \),
\( \{ R = 407.2754233, y_0 = 420.4000000, x_0 = 45.48000000 \} \)
27. \( \{ x_0 = 232.6444444, y_0 = 127.9555556, R = -60.31572078 \} \),
\( \{ R = 60.31572078, x_0 = 232.6444444, y_0 = 127.9555556 \} \)
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled.

```maple
> restart;
> xx := [50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190];
    xx := [50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190]
> yy := [0.8, 0.1, 0, 0, 1, 0.2, 0, 8.1, 7, 3, 4.3, 6, 7.7, 10, 4, 13, 16];
    yy := [0.8, 0.1, 0, 0, 1, 0.2, 0, 8.1, 7, 3, 4.3, 6, 7.7, 10.4, 13, 16]
> n := 15;
```

If we find a circle \( (x - x_0)^2 + (y - y_0)^2 = R^2 \) which passes exactly through the given points \( (x_i, y_i) \), \( i = 1 \ldots n \), then the \( n \) quantities \( (x_i - x_0)^2 + (y_i - y_0)^2 - R^2 \) will all be zero. This is too much to hope for, but we can seek three numbers \( x_0, y_0, R \) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[
F := (x0, y0, R) \rightarrow \sum_{i=1}^{n} ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2
\]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to zero, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).

```maple
> eq1 := diff(F(x0, y0, R), x0) = 0;
    eq1 := 2 ((50 - x0)^2 + (1.8 - y0)^2 - R^2) (-100 + 2 x0)
+ 2 ((60 - x0)^2 + (1.8 - y0)^2 - R^2) (-120 + 2 x0) + 2 ((70 - x0)^2 + y0^2 - R^2) (-140 + 2 x0)
+ 2 ((80 - x0)^2 + y0^2 - R^2) (-160 + 2 x0) + 2 ((90 - x0)^2 + (1 - y0)^2 - R^2) (-180 + 2 x0)
+ 2 ((100 - x0)^2 + (2 - y0)^2 - R^2) (-200 + 2 x0)
+ 2 ((110 - x0)^2 + (8 - y0)^2 - R^2) (-220 + 2 x0)
+ 2 ((120 - x0)^2 + (17 - y0)^2 - R^2) (-240 + 2 x0)
+ 2 ((130 - x0)^2 + (3 - y0)^2 - R^2) (-260 + 2 x0)
+ 2 ((140 - x0)^2 + (4.3 - y0)^2 - R^2) (-280 + 2 x0)
+ 2 ((150 - x0)^2 + (6 - y0)^2 - R^2) (-300 + 2 x0)
+ 2 ((160 - x0)^2 + (7.7 - y0)^2 - R^2) (-320 + 2 x0)
+ 2 ((170 - x0)^2 + (10.4 - y0)^2 - R^2) (-340 + 2 x0)
+ 2 ((180 - x0)^2 + (13 - y0)^2 - R^2) (-360 + 2 x0)
+ 2 ((190 - x0)^2 + (16 - y0)^2 - R^2) (-380 + 2 x0) = 0
> eq2 := diff(F(x0, y0, R), y0) = 0;
    eq2 := 2 ((50 - x0)^2 + (1.8 - y0)^2 - R^2) (-1.6 + 2 y0)
+ 2 ((60 - x0)^2 + (1 - y0)^2 - R^2) (2.7 - 4 ((70 - x0)^2 + y0^2 - R^2) y0)
```
\[ + 4 ((80 - x0)^2 + y0^2 - R^2) y0 + 2 ((90 - x0)^2 + (1 - y0)^2 - R^2) (-2 + 2 y0) + 2 ((100 - x0)^2 + (2 - y0)^2 - R^2) (-4 + 2 y0) + 2 ((110 - x0)^2 + (3 - y0)^2 - R^2) (-1.6 + 2 y0) + 2 ((120 - x0)^2 + (1.7 - y0)^2 - R^2) (-3.4 + 2 y0) + 2 ((130 - x0)^2 + (3 - y0)^2 - R^2) (-6 + 2 y0) + 2 ((140 - x0)^2 + (4.3 - y0)^2 - R^2) (-8.6 + 2 y0) + 2 ((150 - x0)^2 + (6 - y0)^2 - R^2) (-12 + 2 y0) + 2 ((160 - x0)^2 + (7.7 - y0)^2 - R^2) (-15.4 + 2 y0) + 2 ((170 - x0)^2 + (10.4 - y0)^2 - R^2) (-20.8 + 2 y0) + 2 ((180 - x0)^2 + (13 - y0)^2 - R^2) (-26 + 2 y0) + 2 ((190 - x0)^2 + (16 - y0)^2 - R^2) (-32 + 2 y0) = 0 \]

> eq3 := diff(f(x0, y0, R), R) = 0;

\[ eq3 := -4 ((50 - x0)^2 + (1 - y0)^2 - R^2) R - 4 ((60 - x0)^2 + (1 - y0)^2 - R^2) R \]

\[ - 4 ((70 - x0)^2 + y0^2 - R^2) R - 4 ((80 - x0)^2 + y0^2 - R^2) R \]

\[ - 4 ((90 - x0)^2 + (1 - y0)^2 - R^2) R - 4 ((100 - x0)^2 + (2 - y0)^2 - R^2) R \]

\[ - 4 ((110 - x0)^2 + (3 - y0)^2 - R^2) R - 4 ((120 - x0)^2 + (4.3 - y0)^2 - R^2) R \]

\[ - 4 ((130 - x0)^2 + (6 - y0)^2 - R^2) R - 4 ((140 - x0)^2 + (7.7 - y0)^2 - R^2) R \]

\[ - 4 ((150 - x0)^2 + (10.4 - y0)^2 - R^2) R - 4 ((160 - x0)^2 + (13 - y0)^2 - R^2) R \]

\[ - 4 ((170 - x0)^2 + (16 - y0)^2 - R^2) R = 0 \]

> sol := solve({eq1, eq2, eq3}, {x0, y0, R});

\[ sol := \{ R = 0, x0 = 3.270644564 - 7.905740208 I, y0 = 119.9985537 + 74.87152362 I \}, \]

\[ \{ R = 0, y0 = 3.270644564 - 7.905740208 I, x0 = 119.9985537 + 74.87152362 I \}, \]

\[ \{ R = 0, y0 = 6.253808018, x0 = 120.0055250 \}, \]

\[ \{ R = 0, x0 = 120.4229269 - 4.573966278 I, y0 = 3.327996190 + 43.41127291 I \}, \]

\[ \{ R = 0, x0 = 120.4229269 + 4.573966278 I, y0 = 3.327996190 - 43.41127291 I \}, \]

\[ \{ x0 = 81.05143219, R = -376.6705197, y0 = 376.3906171 \}, \]

\[ \{ R = 376.6705197, x0 = 81.05143219, y0 = 376.3906171 \} \]

> circ := (x - x0)^2 + (y - y0)^2 - R^2

\[ circ := (x - x0)^2 + (y - y0)^2 - R^2 \]

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal, contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontourplot, listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto.
pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra-supported,
polyhedrplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot

> p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);

> with(stats[statplots]);
[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift,
yzexchange, zscale, zshift]

> p2 := scatterplot(xx, yy, color=black);
> display([p1, p2], scaling=constrained);
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This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, \[ xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, \\\] \( 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 286 \); \] \( xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, \\\] \( 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 286 \] \] \( yy := [2.8, 1.2, 1.0, 0.9, 0.5, 1.3, 3.8, 5.2, 6.8, 9, 10.8, 13, 16.7, 19.6, 22.1, \\\] \( 25, 28.3, 32, 35.3, 39.1, 44, 47.5, 52.5, 58, 64.2, 74, 80, 86.3, 90 \]; \] \( yy := [2.8, 1.2, 1.0, 0.9, 0.5, 1.3, 3.8, 5.2, 6.8, 9, 10.8, 13, 16.7, 19.6, 22.1, 25, 28.3, 32, 35.3, 39.1, \\\] \( 44, 47.5, 52.5, 58, 64.2, 74, 80, 86.3, 90 \] \] \( n := 30 \); \)

If we find a circle \( (x - x_0)^2 + (y - y_0)^2 = R^2 \) which passes exactly through the given points \([x_i, y_i], \quad i = 1 \ldots n\), then the \( n \) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \( x_0, y_0, \) and \( R \) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \):

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).

\[
equil := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
equil := 2 ((20 - x_0)^2 + (1 - y_0)^2 - R^2) (-40 + 2 x_0)
\]

\[
+ 2 ((30 - x_0)^2 + (9 - y_0)^2 - R^2) (-60 + 2 x_0) + 2 ((40 - x_0)^2 + (13 - y_0)^2 - R^2) (-80 + 2 x_0)
\]

\[
+ 2 ((50 - x_0)^2 + (3 - y_0)^2 - R^2) (-100 + 2 x_0)
\]

\[
+ 2 ((70 - x_0)^2 + (3.8 - y_0)^2 - R^2) (-120 + 2 x_0)
\]

\[
+ 2 ((80 - x_0)^2 + (5.2 - y_0)^2 - R^2) (-140 + 2 x_0)
\]

\[
+ 2 ((90 - x_0)^2 + (6.8 - y_0)^2 - R^2) (-160 + 2 x_0)
\]

\[
+ 2 ((100 - x_0)^2 + (9 - y_0)^2 - R^2) (-200 + 2 x_0)
\]

\[
+ 2 ((110 - x_0)^2 + (10.8 - y_0)^2 - R^2) (-220 + 2 x_0)
\]

\[
+ 2 ((120 - x_0)^2 + (13 - y_0)^2 - R^2) (-240 + 2 x_0)
\]

\[
+ 2 ((130 - x_0)^2 + (16.7 - y_0)^2 - R^2) (-260 + 2 x_0)
\]

\[
+ 2 ((140 - x_0)^2 + (19.6 - y_0)^2 - R^2) (-280 + 2 x_0)
\]

Page 1
\[
+ 2 ((150 - x0)^2 + (22.1 - y0)^2 - R^2) (-300 + 2 x0) \\
+ 2 ((160 - x0)^2 + (25 - y0)^2 - R^2) (-320 + 2 x0) \\
+ 2 ((10 - x0)^2 + (1.2 - y0)^2 - R^1) (-20 + 2 x0) + 4 (x0^2 + (2.8 - y0)^2 - R^2) x0 \\
+ 2 ((170 - x0)^2 + (28.3 - y0)^2 - R^2) (-340 + 2 x0) \\
+ 2 ((180 - x0)^2 + (32 - y0)^2 - R^2) (-360 + 2 x0) \\
+ 2 ((190 - x0)^2 + (35.3 - y0)^2 - R^2) (-380 + 2 x0) \\
+ 2 ((200 - x0)^2 + (39.1 - y0)^2 - R^2) (-400 + 2 x0) \\
+ 2 ((210 - x0)^2 + (44 - y0)^2 - R^2) (-420 + 2 x0) \\
+ 2 ((220 - x0)^2 + (47.5 - y0)^2 - R^2) (-440 + 2 x0) \\
+ 2 ((230 - x0)^2 + (52.5 - y0)^2 - R^2) (-460 + 2 x0) \\
+ 2 ((240 - x0)^2 + (58 - y0)^2 - R^2) (-480 + 2 x0) \\
+ 2 ((250 - x0)^2 + (64.2 - y0)^2 - R^2) (-500 + 2 x0) \\
+ 2 ((260 - x0)^2 + (74 - y0)^2 - R^2) (-520 + 2 x0) \\
+ 2 ((270 - x0)^2 + (80 - y0)^2 - R^2) (-540 + 2 x0) \\
+ 2 ((280 - x0)^2 + (86.3 - y0)^2 - R^2) (-560 + 2 x0) \\
+ 2 ((286 - x0)^2 + (90 - y0)^2 - R^2) (-572 + 2 x0) = 0 \\
\]

\[
eq 2 := d1.\xi \xi (F(x0, y0, R), y0) = 0; \\
eq 2 := 2 (x0^2 + (2.8 - y0)^2 - R^2) (-5.6 + 2 y0) \\
+ 2 ((286 - x0)^2 + (90 - y0)^2 - R^2) (-180 + 2 y0) \\
+ 2 ((170 - x0)^2 + (28.3 - y0)^2 - R^2) (-56.6 + 2 y0) \\
+ 2 ((180 - x0)^2 + (32 - y0)^2 - R^2) (-64 + 2 y0) \\
+ 2 ((190 - x0)^2 + (35.3 - y0)^2 - R^2) (-70.6 + 2 y0) \\
+ 2 ((200 - x0)^2 + (39.1 - y0)^2 - R^2) (-78.2 + 2 y0) \\
+ 2 ((210 - x0)^2 + (44 - y0)^2 - R^2) (-88 + 2 y0) \\
+ 2 ((220 - x0)^2 + (47.5 - y0)^2 - R^2) (-95.0 + 2 y0) \\
+ 2 ((230 - x0)^2 + (52.5 - y0)^2 - R^2) (-105.0 + 2 y0) \\
+ 2 ((240 - x0)^2 + (58 - y0)^2 - R^2) (-116 + 2 y0) \\
+ 2 ((250 - x0)^2 + (64.2 - y0)^2 - R^2) (-128.4 + 2 y0) \\
+ 2 ((260 - x0)^2 + (74 - y0)^2 - R^2) (-148 + 2 y0) \\
+ 2 ((270 - x0)^2 + (80 - y0)^2 - R^2) (-160 + 2 y0) \\
+ 2 ((280 - x0)^2 + (86.3 - y0)^2 - R^2) (-172.6 + 2 y0) \\
+ 2 ((40 - x0)^2 + (5 - y0)^2 - R^2) (-1.0 + 2 y0) \\
+ 2 ((50 - x0)^2 + (1.3 - y0)^2 - R^2) (-2.6 + 2 y0) + 2 ((60 - x0)^2 + (3 - y0)^2 - R^2) (-6 + 2 y0)
\]
\[ +2 \((70-x0)^2 + (3.8-y0)^2 - R^2\) \(\cdot (-7.6 + 2\ x0)\) \\
+2 \((80-x0)^2 + (5.2-y0)^2 - R^2\) \(\cdot (-10.4 + 2\ y0)\) \\
+2 \((90-x0)^2 + (6.8-y0)^2 - R^2\) \(\cdot (-13.6 + 2\ y0)\) \\
+2 \((100-x0)^2 + (9-y0)^2 - R^2\) \(\cdot (-18 + 2\ y0)\) \\
+2 \((110-x0)^2 + (10.8-y0)^2 - R^2\) \(\cdot (-21.6 + 2\ y0)\) \\
+2 \((120-x0)^2 + (13-y0)^2 - R^2\) \(\cdot (-26 + 2\ y0)\) \\
+2 \((130-x0)^2 + (16.7-y0)^2 - R^2\) \(\cdot (-33.4 + 2\ y0)\) \\
+2 \((140-x0)^2 + (19.6-y0)^2 - R^2\) \(\cdot (-39.2 + 2\ y0)\) \\
+2 \((150-x0)^2 + (22.1-y0)^2 - R^2\) \(\cdot (-44.2 + 2\ y0)\) \\
+2 \((160-x0)^2 + (25-y0)^2 - R^2\) \(\cdot (-50 + 2\ y0)\) \\
+2 \((170-x0)^2 + (1.2-y0)^2 - R^2\) \(\cdot (-2.4 + 2\ y0)\) \\
+2 \((20-x0)^2 + (1-y0)^2 - R^2\) \(\cdot (-2 + 2\ y0)\) \\
+2 \((30-x0)^2 + (9-y0)^2 - R^2\) \(\cdot (-1.8 + 2\ y0)\) = 0 \]

\[ eq3 := \text{diff}(F(x0, y0, R), R) = 0; \]

\[ eq3 := -4 \((10-x0)^2 + (1.2-y0)^2 - R^2\) R - 4 \((20-x0)^2 + (1-y0)^2 - R^2\) R - 4 \((30-x0)^2 + (9-y0)^2 - R^2\) R - 4 \((50-x0)^2 + (1.3-y0)^2 - R^2\) R - 4 \((70-x0)^2 + (3.8-y0)^2 - R^2\) R - 4 \((90-x0)^2 + (6.8-y0)^2 - R^2\) R - 4 \((110-x0)^2 + (10.8-y0)^2 - R^2\) R - 4 \((130-x0)^2 + (16.7-y0)^2 - R^2\) R - 4 \((150-x0)^2 + (22.1-y0)^2 - R^2\) R - 4 \((160-x0)^2 + (25-y0)^2 - R^2\) R - 4 \((170-x0)^2 + (28.3-y0)^2 - R^2\) R - 4 \((180-x0)^2 + (32-y0)^2 - R^2\) R - 4 \((190-x0)^2 + (35.3-y0)^2 - R^2\) R - 4 \((200-x0)^2 + (39.1-y0)^2 - R^2\) R - 4 \((210-x0)^2 + (44-y0)^2 - R^2\) R - 4 \((220-x0)^2 + (47.5-y0)^2 - R^2\) R - 4 \((230-x0)^2 + (52.5-y0)^2 - R^2\) R - 4 \((240-x0)^2 + (58-y0)^2 - R^2\) R - 4 \((250-x0)^2 + (64.2-y0)^2 - R^2\) R - 4 \((260-x0)^2 + (74-y0)^2 - R^2\) R - 4 \((270-x0)^2 + (80-y0)^2 - R^2\) R - 4 \((280-x0)^2 + (86.3-y0)^2 - R^2\) R - 4 \((286-x0)^2 + (90-y0)^2 - R^2\) R = 0 \]

\[ sol := \text{solve}([eq1, eq2, eq3], [x0, y0, R]); \]

\[ sol := [R = 0, x0 = 144.6920193, y0 = 36.60446776]; \]

\[ \{ R = 0, x0 = 144.9974914 \pm 149.7161955 I, y0 = 25.25272283 \pm 46.35674453 I \}; \]

\[ \{ R = 0, x0 = 144.9974914 \pm 149.7161955 I, y0 = 25.25272283 \pm 46.35674453 I \}; \]

\[ \{ R = 0, y0 = 26.60854162 + 87.68554408 I, x0 = 148.8937030 - 26.97214513 I \}; \]

\[ \{ R = 0, x0 = 148.8937030 + 26.97214513 I, y0 = 26.60854162 - 87.68554408 I \}; \]
\{ y_0 = 445.9756616, x_0 = 19.23446758, R = -444.7160848 \}.
\{ y_0 = 445.9756616, x_0 = 19.23446758, R = 444.7160848 \}

> \text{circ} := (x-x_0)^2 + (y-y_0)^2 = R^2;

\[ \text{circ} := (x-x_0)^2 + (y-y_0)^2 = R^2 \]

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
 contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
 fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontourplot,
 listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
 pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
 polyhedrplot, replot, rootplot, semilogplot, setoptions, setoptions3d, spacecurve,
 sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

> p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);

> with(stats[statplots]);

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift,
 yexchange, zscale, zshift]

> p2 := scatterplot(xx, yy, color=black);

> display([p1, p2], scaling=constrained);
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled.

```latex
\texttt{restart;}
\texttt{xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270];}
\texttt{xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270];}
\texttt{yy := [2.8, 9, 3, 6.8, 13, 22.1, 32, 44, 58, 80];}
\texttt{yy := [2.8, 9, 3, 6.8, 13, 22.1, 32, 44, 58, 80];}
\texttt{n := 10;}
```

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\) , \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \) and \(R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum \left( (x_i - x_0)^2 + (y_i - y_0)^2 - R^2 \right)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, \) and \(R\).

```latex
\texttt{eq1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;}
\texttt{eq1 := 4 (x_0 + (2.8 - y_0)^2 - R^2) x_0 + 2 ((30 - x_0)^2 + (.9 - y_0)^2 - R^2) (-60 + 2 x_0) + 2 ((60 - x_0)^2 + (3 - y_0)^2 - R^2) (-120 + 2 x_0) + 2 ((90 - x_0)^2 + (6.8 - y_0)^2 - R^2) (-180 + 2 x_0) + 2 ((120 - x_0)^2 + (13 - y_0)^2 - R^2) (-240 + 2 x_0) + 2 ((150 - x_0)^2 + (22.1 - y_0)^2 - R^2) (-300 + 2 x_0) + 2 ((180 - x_0)^2 + (32 - y_0)^2 - R^2) (-360 + 2 x_0) + 2 ((210 - x_0)^2 + (44 - y_0)^2 - R^2) (-420 + 2 x_0) + 2 ((240 - x_0)^2 + (58 - y_0)^2 - R^2) (-480 + 2 x_0) + 2 ((270 - x_0)^2 + (80 - y_0)^2 - R^2) (-540 + 2 x_0) = 0}
```

```latex
\texttt{eq2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;}
\texttt{eq2 := 2 (x_0 + (2.8 - y_0)^2 - R^2) (-5.6 + 2 y_0) + 2 ((30 - x_0)^2 + (.9 - y_0)^2 - R^2) (-1.8 + 2 y_0) + 2 ((60 - x_0)^2 + (3 - y_0)^2 - R^2) (-6 + 2 y_0) + 2 ((90 - x_0)^2 + (6.8 - y_0)^2 - R^2) (-13.6 + 2 y_0) + 2 ((120 - x_0)^2 + (13 - y_0)^2 - R^2) (-26 + 2 y_0) + 2 ((150 - x_0)^2 + (22.1 - y_0)^2 - R^2) (-44.2 + 2 y_0) + 2 ((180 - x_0)^2 + (32 - y_0)^2 - R^2) (-64 + 2 y_0) = 0}
```

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\[+2((210-x0)^2+(44-y0)^2-R^2)(-88+2y0)\]
\[+2((240-x0)^2+(58-y0)^2-R^2)(-116+2y0)\]
\[+2((270-x0)^2+(80-y0)^2-R^2)(-160+2y0)\]
\[=0\]

\[eq3:=diff(F(x0,y0,R),R)\times0;\]

\[eq3:=-4((x0^2+(2.8-y0)^2-R^2)\times R - 4((30-x0)^2+(.9-y0)^2-R^2)\times R\]
\[-4((60-x0)^2+(3-y0)^2-R^2)\times R - 4((90-x0)^2+(6.8-y0)^2-R^2)\times R\]
\[-4((120-x0)^2+(13-y0)^2-R^2)\times R - 4((150-x0)^2+(22.1-y0)^2-R^2)\times R\]
\[-4((180-x0)^2+(32-y0)^2-R^2)\times R - 4((210-x0)^2+(44.8-y0)^2-R^2)\times R\]
\[-4((240-x0)^2+(58-y0)^2-R^2)\times R - 4((270-x0)^2+(80-y0)^2-R^2)\times R\]
\[=0\]

\[sol:=\text{solve}\{eq1,eq2,eq3\},(x0,y0,R)\};\]

\[sol:=\{R=0,x0=135.0143055-149.4426956i,y0=22.42085089-42.25846479i\},\]
\[\{R=0,x0=22.42085089+42.25846479i,y0=135.0143055+149.4426956i\},\]
\[\{R=0,x0=33.65783526,x0=135.059759i\},\]
\[\{R=23.68915098+87.53032781i,x0=138.9784870-24.57348938i\},\]
\[\{R=23.68915098+24.57348938i,x0=138.9784870-87.53032781i\},\]
\[\{y0=437.6686732,x0=22.10277830,R=-435.9848867\},\]
\[\{y0=437.6686732,x0=22.10277830,R=435.9848867\}\]

\[\text{circ}:=(x-x0)^2+(y-y0)^2=R^2\]

\[\text{with}\{\text{plots}\};\]

\(\text{animate},\text{animate3d},\text{animatecurve},\text{changecoords},\text{complexplot},\text{complexplot3d},\text{conformal},\]
\(\text{contourplot},\text{contourplot3d},\text{coordplot},\text{coordplot3d},\text{cylinderplot},\text{densityplot},\text{display},\text{display3d},\]
\(\text{fieldplot},\text{fieldplot3d},\text{gradplot},\text{gradplot3d},\text{implicitplot},\text{implicitplot3d},\text{inequal},\text{listcontplot},\]
\(\text{listcontplot3d},\text{listdensityplot},\text{listplot},\text{listplot3d},\text{loglogplot},\text{logplot},\text{matrixplot},\text{odeplot},\text{pawax},\]
\(\text{poinplot},\text{pointplot3d},\text{polargraph},\text{polygonplot},\text{polygonplot3d},\text{polyhedra\_supported},\]
\(\text{polyhedrplot},\text{replot},\text{rootof},\text{semilogplot},\text{setoptions},\text{setoptions3d},\text{spacecurve},\]
\(\text{sparsematrixplot},\text{sphereplot},\text{sufdata},\text{textplot},\text{textplot3d},\text{tubeplot}\}\]

\[\text{pl1:=implicitplot}(\text{subs}(\text{sol}[7]),\text{circ},x=0..400,y=-300..300,\text{numpoints}=4000);\]

\[\text{with}\{\text{stats}[,\text{statplots}]\};\]

\(\text{boxplot},\text{histogram},\text{scatterplot},\text{xscale},\text{xshift},\text{xyexchange},\text{xxexchange},\text{yscale},\text{yshift},\]
\(\text{yexchange},\text{zscale},\text{zshift}\}\]

\[\text{p2:=scatterplot}(\text{xx},\text{yy},\text{color=black});\]

\[\text{display}([\text{pl1, p2}], \text{scaling=constrained});\]
$R = 4.36 \, \text{m}$
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list $xx$, and the corresponding y coordinates are given as a list $yy$. The number of data points is $n$. The data I entered was labelled.

```maple
restart;
xx := [50, 100, 150, 200, 250];

yy := [1.3, 9, 22.1, 39.1, 64.2];
n := 5;
```

If we find a circle $(x - x_0)^2 + (y - y_0)^2 = R^2$ which passes exactly through the given points $[x_i, y_i]$, $i = 1 \ldots n$, then the $n$ quantities $(x_i - x_0)^2 + (y_i - y_0)^2 - R^2$ will all be zero. This is too much to hope for, but we can seek three numbers $x_0, y_0, R$ so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function $F$.

```maple
F := \sum ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2, i = 1 \ldots n);
```

We seek a critical point for the function $F$ by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns $x_0, y_0, R$.

```maple
eq1 := \text{diff}(F(x0, y0, R), x0) = 0;
eq1 := 2 ((50 \cdot x0)^2 + (1.3 - y0)^2 - R^2) (-100 + 2 x0) + 2 ((100 - x0)^2 + (9 - y0)^2 - R^2) (-200 + 2 x0) + 2 ((150 - x0)^2 + (22.1 - y0)^2 - R^2) (-300 + 2 x0) + 2 ((200 - x0)^2 + (39.1 - y0)^2 - R^2) (-400 + 2 x0) + 2 ((250 - x0)^2 + (64.2 - y0)^2 - R^2) (-500 + 2 x0) = 0;
eq2 := \text{diff}(F(x0, y0, R), y0) = 0;
eq2 := 2 ((50 \cdot x0)^2 + (1.3 - y0)^2 - R^2) (-2.6 + 2 y0) + 2 ((100 - x0)^2 + (9 - y0)^2 - R^2) (-18 + 2 y0) + 2 ((150 - x0)^2 + (22.1 - y0)^2 - R^2) (-44.2 + 2 y0) + 2 ((200 - x0)^2 + (39.1 - y0)^2 - R^2) (-78.2 + 2 y0) + 2 ((250 - x0)^2 + (64.2 - y0)^2 - R^2) (-128.4 + 2 y0) = 0;
eq3 := \text{diff}(F(x0, y0, R), R) = 0;
eq3 := -4 ((50 \cdot x0)^2 + (1.3 - y0)^2 - R^2) R - 4 ((100 - x0)^2 + (9 - y0)^2 - R^2) R - 4 ((150 - x0)^2 + (22.1 - y0)^2 - R^2) R - 4 ((200 - x0)^2 + (39.1 - y0)^2 - R^2) R - 4 ((250 - x0)^2 + (64.2 - y0)^2 - R^2) R = 0;
sol := \text{solve}([eq1, eq2, eq3], [x0, y0, R]);
```

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sol := \{ R = 0, x0 = 150.0008615 - 122.5352909 \textit{i}, y0 = 25.18714739 - 38.38444667 \textit{i} \};
\{ R = 0, x0 = 25.18714739 + 38.38444667 \textit{i}, x0 = 150.0008615 + 122.5352909 \textit{i} \};
\{ R = 0, x0 = 150.0167878, y0 = 30.98669374 \},
\{ R = 0, y0 = 25.91535389 + 71.17553811 \textit{i}, x0 = 152.2251910 - 22.2341220 \textit{i} \};
\{ R = 0, y0 = 25.91535389 - 71.17553811 \textit{i}, x0 = 152.2251910 + 22.2341220 \textit{i} \};
\{ x0 = -2.849978627, y0 = 521.2882894, R = -522,5450969 \},
\{ x0 = -2.849978627, y0 = 521.2882894, R = 522,5450969 \}
\texttt{circ:=(x-x0)^2+(y-y0)^2=R^2};
\texttt{circ:=(x-x0)^2+(y-y0)^2=R^2};
\texttt{with(plots)};
\texttt{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,}
\texttt{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,}
\texttt{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontourplot,}
\texttt{listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,}
\texttt{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra\_supported,}
\texttt{polyhedrplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,}
\texttt{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot}\}
\texttt{pl:=implicitplot(subs(sol[7],circ)),x=-.400,y=-300..300,numpoints=4000);}
\texttt{with(stats[statplots]);}
\texttt{boxplot, histogram, scatterplot, xscale, xshift, yxexchange, xzexchange, yscale, yshift,}
\texttt{yxexchange, zscale, zshift}\}
\texttt{p2:=scatterplot(xx,yy,color=black);}
\texttt{display([pl,p2],scaling=constrained);}
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, \[ xx := [50, 150, 250] \]

\[ yy := [1.3, 22.1, 64.2] \]

\[ n := 3 \]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly through the given points \((x_i, y_i)\), \(i = 1 \ldots n\), then the \( n \) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, \) and \( R \) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \( F \).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \( F \) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \( x_0, y_0, \) and \( R \).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]

\[
eq 3 := \text{diff}(F(x_0, y_0, R), R) = 0;
\]

\[
sol := \text{solve}\{\{\text{eq1}, \text{eq2}, \text{eq3}\}, \{x_0, y_0, R\}\};
\]

\[
sol := \{ R = 0, x_0 = 150.0019101 - 141.4794573 I, y_0 = 27.42808975 - 44.66799123 I \},
\]

\[
\{ R = 0, x_0 = 150.0019101 + 141.4794573 I, y_0 = 27.42808975 + 44.66799123 I \},
\]

\[
\{ R = 0, y_0 = 32.69908417, x_0 = 150.0103159 \},
\]

\[
\{ R = 0, y_0 = 28.09225155 + 28.0907192 I, x_0 = 152.0293746 - 25.87311945 I \},
\]

\[
\{ R = 0, x_0 = 152.0293746 + 25.87311945 I, y_0 = 28.09225155 - 28.0907192 I \},
\]

\[
\{ x_0 = -10.58225347, y_0 = 543.3453052, R = 545.4203138 \}.
\]
\[
\{ x_0 = -10.58222347, y_0 = 543.3453052, R = 545.4203138 \}
\]

\[
circ := (x - x0)^2 + (y - y0)^2 = R^2
\]

\[
\text{with(pplots);}
\]

\[
\text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,}
\text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,}
\text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,}
\text{listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,}
\text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,}
\text{polyhedralplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,}
\text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot}
\]

\[
\text{p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000)};
\]

\[
\text{with(stats[statplots]);}
\]

\[
\text{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,}
\text{yexchange, zscale, zshift}
\]

\[
p2 := \text{scatterplot}(\text{xx, yy, color=black)};
\]

\[
\text{display([p1, p2], scaling=constrained)};
\]
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx, and the list of y coordinates as yy. The number of points in the list is n. The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order. There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "Futlock VE, BIE".

```plaintext
> restart;

> xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 280, 286];
> yy := [2.8, 1.2, 1.9, 0.5, 1.3, 3.8, 5.2, 6.8, 9.1, 10.8, 13.6, 16.7, 19.6, 22.1, 25, 28.3, 32, 35.3, 39.1, 44, 47.5, 52.5, 58, 64.2, 74, 80, 86.3, 90];

> cir := (x, y) -> ((x-x0)^2 + (y-y0)^2 = R^2;
> cir := (x, y) -> (x-x0)^2 + (y-y0)^2 = R^2

> n := 30;

> for i from 1 to n-2 do
> eq1 := cir(xx[i], yy[i]);
> eq2 := cir(xx[i+1], yy[i+1]);
> eq3 := cir(xx[i+2], yy[i+2]);
> print(i, solve({eq1, eq2, eq3}, {x0, y0, R}));
> end;
```

1. \( R = -72.64397818, x0 = 16.44914286, y0 = 73.55714286 \)
\( x0 = 16.44914286, y0 = 73.55714286, R = 72.64397818 \)
2. \( R = 1001.250000, x0 = -1000.362511, y0 = 35.00300000 \)
\( x0 = 1001.250000, y0 = 35.00300000, R = 1000.362511 \)
3. \( R = -333.7208119, x0 = -332.7166667, y0 = 21.66333333 \)
\( x0 = -333.7208119, y0 = -332.7166667, x0 = 21.66333333 \)
4. \( R = -83.68315924, y0 = 84.16666667, x0 = 38.33866667 \)
\( y0 = 84.16666667, x0 = 38.33866667, R = 83.68315924 \)
5. \( x0 = 35.92222222, y0 = 114.37222222, R = -113.9452117 \)
\( x0 = 35.92222222, y0 = 114.37222222, R = -113.9452117 \)
6. \( x0 = -113.9452117, y0 = -110.07222222, R = 74.07777778 \)
\( x0 = -113.9452117, y0 = -110.07222222, x0 = 74.07777778 \)
7. \( y0 = 172.6333333, x0 = 51.46133333, R = -169.8480986 \)

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\{ y_0 = 172.6333333, x_0 = 51.46133333, R = 169.8480986 \}
8. \{ R = -517.0179034, y_0 = 516.5000000, x_0 = 3.320000000 \},
   \{ R = 517.0179034, y_0 = 516.5000000, x_0 = 3.320000000 \}
9. \{ x_0 = 57.21866667, y_0 = 179.6333333, R = -175.9146865 \},
   \{ x_0 = 57.21866667, y_0 = 179.6333333, R = 175.9146865 \}
10. \{ x_0 = 151.9800000, y_0 = -251.1000000, R = -265.2431534 \},
    \{ x_0 = 151.9800000, y_0 = -251.1000000, R = 265.2431534 \}
11. \{ R = -265.2431534, y_0 = 270.9000000, x_0 = 58.0200000 \},
    \{ R = 265.2431534, y_0 = 270.9000000, x_0 = 58.0200000 \}
12. \{ x_0 = 98.73246667, y_0 = 85.84333333, R = -75.88451216 \},
    \{ x_0 = 98.73246667, y_0 = 85.84333333, R = 75.88451216 \}
13. \{ R = -146.1342484, y_0 = -122.1125000, x_0 = 175.6761250 \},
    \{ y_0 = -122.1125000, x_0 = 175.6761250, R = 146.1342484 \}
14. \{ x_0 = 212.3937500, y_0 = -248.7250000, R = -277.9193420 \},
    \{ x_0 = 212.3937500, y_0 = -248.7250000, R = 277.9193420 \}
15. \{ x_0 = 77.6062500, y_0 = 290.4250000, R = -277.9193420 \},
    \{ x_0 = 77.6062500, y_0 = 290.4250000, R = 277.9193420 \}
16. \{ x_0 = 75.0832500, y_0 = 299.1250000, R = -286.9762535 \},
    \{ x_0 = 75.0832500, y_0 = 299.1250000, R = 286.9762535 \}
17. \{ y_0 = 309.0250000, x_0 = 71.81625000, R = -297.3996879 \},
    \{ y_0 = 309.0250000, x_0 = 71.81625000, R = 297.3996879 \}
18. \{ R = -297.3996879, x_0 = 278.1837500, y_0 = -248.7250000 \},
    \{ R = 297.3996879, x_0 = 278.1837500, y_0 = -248.7250000 \}
19. \{ R = -239.0777326, x_0 = 110.0966000, y_0 = 260.6300000 \},
    \{ R = 239.0777326, x_0 = 110.0966000, y_0 = 260.6300000 \}
20. \{ R = -118.1018249, x_0 = 153.0911818, y_0 = 147.4863636 \},
    \{ x_0 = 153.0911818, y_0 = 147.4863636, R = 118.1018249 \}
21. \{ y_0 = 40.37857143, R = -91.40522054, x_0 = 245.1450000 \},
    \{ y_0 = -40.37857143, x_0 = 245.1450000, R = 91.40522054 \}
22. \{ y_0 = 126.5833333, R = -85.80506209, x_0 = 186.7083333 \},
    \{ R = 85.80506209, y_0 = 126.5833333, x_0 = 186.7083333 \}
23. \{ x_0 = 96.1250000, y_0 = 307.7500000, R = -288.2274764 \},
    \{ x_0 = 96.1250000, y_0 = 307.7500000, R = 288.2274764 \}
24. \{ x_0 = 127.9307143, y_0 = 249.9214286, R = -222.2461688 \},
    \{ x_0 = 127.9307143, y_0 = 249.9214286, R = 222.2461688 \}
25. \{ x_0 = 214.2755556, y_0 = 110.6555556, R = -58.60336657 \}

\{ x_0 = 214.2755556, y_0 = 110.6555556, R = 58.60336657 \}

\{ R = -54.76014478, y_0 = 30.31052632, x_0 = 293.0136842 \},
\{ R = 54.76014478, y_0 = 30.31052632, x_0 = 293.0136842 \}

\{ x_0 = -12.49000000, R = -539.3751326, y_0 = 539.4833333 \},
\{ x_0 = -12.49000000, R = 539.3751326, y_0 = 539.4833333 \}

\{ x_0 = 798.6875000, y_0 = -748.1000000, R = -982.4765049 \},
\{ R = 982.4765049, x_0 = 798.6875000, y_0 = -748.1000000 \}
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx, and the list of y coordinates as yy. The number of points in the list is n. The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "FuTock V. B7E".

```plaintext
> restart;
> xx := [0, 30, 60, 90, 120, 150, 180, 210, 240, 270];
> yy := [2.8, .9, 3, 6.8, 13, 22.1, 32, 44, 58, 80];
> cir := (x, y) -> (x-x0)^2 + (y-y0)^2 = R^2;
> n := 10;
> n := 10
> for i from 1 to n-2 do
>   eq1 := cir(xx[i], yy[i]);
>   eq2 := cir(xx[i+1], yy[i+1]);
>   eq3 := cir(xx[i+2], yy[i+2]);
>   print(i, solve({eq1, eq2, eq3}, {x0, y0, R}));
> end do;
```

1. \{ y0 = 226.9025000, x0 = 29.25332500, R = -226.0037334 \}
2. \{ y0 = 226.9025000, x0 = 29.25332500, R = 226.0037334 \}
3. \{ y0 = 537.9558824, x0 = 7.479588235, R = -537.5278502 \}
4. \{ y0 = 537.9558824, x0 = 7.479588235, R = 537.5278502 \}
5. \{ y0 = 392.8166667, x0 = 25.86388889, R = -391.3084559 \}
6. \{ y0 = 392.8166667, x0 = 25.86388889, R = 391.3084559 \}
7. \{ x0 = \frac{-107}{5}, y0 = 579, R = \frac{1}{5} \cdot \text{RootOf}(-Z^2 - 8494274) \}
8. \{ R = \frac{1}{5} \cdot \text{RootOf}(-Z^2 - 805834), x0 = \frac{747}{5}, y0 = 213 \}
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx, and the list of y coordinates as yy. The number of points in the list is n. The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i, and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled "Futock VI, B6E".

```
> restart;
> xx := [3, 30, 60, 90, 120, 150, 180, 210, 240, 270];
> yy := [0, 7, 1, 4.8, 12.6, 22.1, 33.3, 46.7, 62.8, 80.6];
> cir := (x, y) -> ((x-x0)^2+(y-y0)^2)=R^2;
> n := 10;  

> for i from 1 to n-2 do
>  eq1 := cir(xx[i], yy[i]);
>  eq2 := cir(xx[i+1], yy[i+1]);
>  eq3 := cir(xx[i+2], yy[i+2]);
>  print(i, solve({eq1, eq2, eq3}, {x0, y0, R}));
> end:
```

1. \[ x0 = 62.90348837, y0 = -1789.498837, R = -1790.501191 \]
   \[ x0 = 62.90348837, y0 = -1789.498837, R = 1790.501191 \]
2. \[ x0 = 42.40631429, y0 = 260.2185714, R = -259.8149448 \]
   \[ x0 = 42.40631429, y0 = 260.2185714, R = 259.8149448 \]
3. \[ y0 = 239.2100000, x0 = 45.06740000, R = -238.6775788 \]
   \[ y0 = 239.2100000, x0 = 45.06740000, R = 238.6775788 \]
4. \[ R = -597.1597933, x0 = -45.21500000, y0 = 586.4500000 \]
   \[ R = 597.1597933, x0 = -45.21500000, y0 = 586.4500000 \]
5. \[ y0 = 614.9500000, x0 = -54.24000000, R = -627.0447353 \]
   \[ y0 = 614.9500000, x0 = -54.24000000, R = 627.0447353 \]
6. \[ y0 = 511.7090909, x0 = -15.69672727, R = -516.8872869 \]
   \[ y0 = 511.7090909, x0 = -15.69672727, R = 516.8872869 \]
7. \[ R = 461.6951827, y0 = 461.2870370, x0 = 6.825123457 \]
   \[ R = 461.6951827, y0 = 461.2870370, x0 = 6.825123457 \]
8. \[ x0 = -154.3633529, y0 = 761.6382353, R = 802.4321362 \]
   \[ x0 = -154.3633529, y0 = 761.6382353, R = 802.4321362 \]
B8E
Port side
Futtock IIII
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list \( xx \), and the corresponding y coordinates are given as a list \( yy \). The number of data points is \( n \). The data I entered was labelled, 

\[
\begin{align*}
\text{restart;} & \\
xx & := [10, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, \\
& \quad 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 276]; \\
\text{yy} & := [13.8, 2.9, 2.5, 2.1, 3, 1.6, 2, 1.8, 1.2, 1.5, 1.2, 3.7, 5, 6, 3, 8, 9, 10.7, 12, 14.3, 16.8, 19, 21.5, 23.7, 27.2, \\
& \quad 30.5, 33.5, 37, 40.5, 44, 47.5, 50]; \\
n & := 29;
\end{align*}
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\): 

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((xx[i] - x0)^2 + (yy[i] - y0)^2 - R^2)^2, \quad i = 1 \ldots n;
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\). 

\[
eq \text{eq}1 := \text{diff}(F(x_0, y_0, R), x0) = 0;
\]

\[
eq \text{eq}2 := 2 \left( (230 - x0)^2 + (33.5 - y0)^2 - R^2 \right) (-460 + 2 x0) \\
- 2 \left( (240 - x0)^2 + (37 - y0)^2 - R^2 \right) (-480 + 2 x0) \\
+ 2 \left( (250 - x0)^2 + (40.5 - y0)^2 - R^2 \right) (-500 + 2 x0) \\
+ 2 \left( (260 - x0)^2 + (44 - y0)^2 - R^2 \right) (-520 + 2 x0) \\
+ 2 \left( (270 - x0)^2 + (47.5 - y0)^2 - R^2 \right) (-540 + 2 x0) \\
+ 2 \left( (276 - x0)^2 + (50 - y0)^2 - R^2 \right) (-532 + 2 x0) \\
+ 2 \left( (170 - x0)^2 + (16.8 - y0)^2 - R^2 \right) (-340 + 2 x0) + 4 (x0^2 + (3.8 - y0)^2 - R^2) x0 \\
+ 2 \left( (10 - x0)^2 + (2.9 - y0)^2 - R^2 \right) (-20 + 2 x0) \\
+ 2 \left( (20 - x0)^2 + (2.5 - y0)^2 - R^2 \right) (-40 + 2 x0) \\
+ 2 \left( (190 - x0)^2 + (21.5 - y0)^2 - R^2 \right) (-380 + 2 x0) \\
+ 2 \left( (200 - x0)^2 + (23.7 - y0)^2 - R^2 \right) (-400 + 2 x0) \\
+ 2 \left( (210 - x0)^2 + (27.2 - y0)^2 - R^2 \right) (-420 + 2 x0)
\]
\[ +2 \left( (20 - x_0)^2 + (2.5 - y_0)^2 - R^2 \right) (-5.0 + 2 y_0) \\
+2 \left( (190 - x_0)^2 + (21.5 - y_0)^2 - R^2 \right) (-43.0 + 2 y_0) \\
+2 \left( (200 - x_0)^2 + (23.7 - y_0)^2 - R^2 \right) (-47.4 + 2 y_0) \\
+2 \left( (210 - x_0)^2 + (27.2 - y_0)^2 - R^2 \right) (-54.4 + 2 y_0) \\
+2 \left( (220 - x_0)^2 + (30.5 - y_0)^2 - R^2 \right) (-61.0 + 2 y_0) \\
+2 \left( (230 - x_0)^2 + (33.5 - y_0)^2 - R^2 \right) (-67.0 + 2 y_0) \\
+2 \left( (240 - x_0)^2 + (37 - y_0)^2 - R^2 \right) (-74 + 2 y_0) \\
+2 \left( (250 - x_0)^2 + (40.5 - y_0)^2 - R^2 \right) (-81.0 + 2 y_0) \\
+2 \left( (260 - x_0)^2 + (44 - y_0)^2 - R^2 \right) (-88 + 2 y_0) \\
+2 \left( (270 - x_0)^2 + (47.5 - y_0)^2 - R^2 \right) (-95.0 + 2 y_0) \\
+2 \left( (276 - x_0)^2 + (50 - y_0)^2 - R^2 \right) (-100 + 2 y_0) = 0 \\
\]

\[ \text{eq3 := diff(F(x_0,y_0,R),R) = 0;} \]

\[ \text{sol := solve}\{\text{eq1, eq2, eq3}\}, (x_0, y_0, R)\}; \]

\[ \text{sol := \{ R = 0, y_0 = 14.10417726 - 25.32629391 i, x_0 = 139.9920491 - 144.6244892 i \},} \]

\[ \text{sol := \{ R = 0, y_0 = 14.10417726 + 25.32629391 i, x_0 = 139.9920491 + 144.6244892 i \},} \]

\[ \text{sol := \{ R = 0, y_0 = 14.34584077 + 84.03856120 i, x_0 = 141.1127005 + 146.8399721 i \},} \]

\[ \text{sol := \{ R = 0, x_0 = 141.1127005 - 146.8399721 i, y_0 = 14.34584077 + 84.03856120 i \},} \]

\[ \text{sol := \{ R = 0, y_0 = 21.20308638, x_0 = 139.6176339 \},} \]

\[ \text{y_0 = 596.9818487, x_0 = 39.13647928, R = -595.2377670 \}.} \]
(R = 595.2377670, y0 = 596.9818487, x0 = 39.13647928)

> circ := (x-x0)^2 + (y-y0)^2 = R^2;

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
 contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
 fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
 listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
 pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra_supported,
 polyhedrplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
 sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]

> p1 := implicitplot(subs(sol[y], circ), x=0..400, y=-300..300, numpoints = 4900);

> with(stats[statplots]);

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,
 yexchange, zscale, zshift]

> p2 := scatterplot(xx, yy, color=black);

> display([p1, p2], scaling=constrained);
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The data I entered was labelled: xx := [30, 60, 90, 120, 150, 180, 210, 240, 270]

yy := [2, 5, 3, 7, 8, 12, 19, 27.2, 37.4, 51]

n := 9;

If we find a circle \((x-x_0)^2 + (y-y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i-x_0)^2 + (y_i-y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\) so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} \left( (x_i-x_0)^2 + (y_i-y_0)^2 - R^2 \right)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]

\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]
\[ +2 ((210 - x0)^2 + (27.2 - y0)^2 - R^2) (-54.4 + 2 y0) + 2 ((240 - x0)^2 + (37 - y0)^2 - R^2) (-74 + 2 y0) + 2 ((270 - x0)^2 + (47.5 - y0)^2 - R^2) (-95.0 + 2 y0) = 0 \]

\[ \text{eq3 :=} \text{diff}(R(x0, y0, R), R) = 0; \]

\[ \text{eq3 :=} -4 ((30 - x0)^2 + (2 - y0)^2 - R^2) R - 4 ((90 - x0)^2 + (3.7 - y0)^2 - R^2) R - 4 ((150 - x0)^2 + (12 - y0)^2 - R^2) R - 4 ((180 - x0)^2 + (19 - y0)^2 - R^2) R - 4 ((210 - x0)^2 + (27.2 - y0)^2 - R^2) R - 4 ((240 - x0)^2 + (37 - y0)^2 - R^2) R - 4 ((270 - x0)^2 + (47.5 - y0)^2 - R^2) R = 0 \]

\[ \text{sol := solve}([\text{eq1, eq2, eq3}], \{x0, y0, R\}); \]

\[ \text{sol :=} \{ R = 0, x0 = 149.9972670, y0 = 21.23867041 \}, \]

\[ \{ R = 0, x0 = 150.0067224 - 134.2373912 l, y0 = 15.47911920 - 26.16465698 l \}, \]

\[ \{ R = 0, x0 = 150.0067224 + 134.2373912 l, y0 = 15.47911920 + 26.16465698 l \}, \]

\[ \{ R = 0, x0 = 151.4478996 - 15.14626591 l, y0 = 15.78817238 + 77.89472883 l \}, \]

\[ \{ R = 0, x0 = 151.4478996 + 15.14626591 l, y0 = 15.78817238 - 77.89472883 l \}, \]

\[ y0 = 621.5500272, x0 = 33.44474472, R = 620.3131562 \}, \]

\[ \{ y0 = 621.5500272, x0 = 33.44474472, R = 620.3131562 \} \]

\[ \text{circ :=} (x-x0)^2 + (y-y0)^2 = R^2; \]

\[ \text{with}(\text{plots}); \]

\[ \{ \text{animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,} \]

\[ \text{contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,} \]

\[ \text{fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontourplot,} \]

\[ \text{listcontourplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,} \]

\[ \text{pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra, supported,} \]

\[ \text{polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,} \]

\[ \text{sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot} \]

\[ \text{p1 := implicitplot}([\text{subs(sol[7], circ)}], x=0..400, y=-300..300, numpoints=4000); \]

\[ \text{with}([\text{stats}[\text{statplots}]);} \]

\[ \{ \text{boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xzexchange, yscale, yshift,} \]

\[ \text{xyexchange, zscale, zshift} \]

\[ \text{p2 := scatterplot(xx, yy, color=black);} \]

\[ \text{display}([\text{p1, p2}, scaling=constrained]); \]
This worksheet is to find a circle which best fits a list of data points. The x coordinates are given as a list xx, and the corresponding y coordinates are given as a list yy. The number of data points is n. The data I entered was labelled.

> restart;

> xx := [50, 100, 150, 200, 250];

> yy := [1.2, 5, 12, 23.7, 40.5];

> n := 5;

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\)

for \(i = 1..n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to

hope for, but we can seek three numbers \(x_0, y_0,\) and \(R\) so that the sum of the squares of these

quantities is as small as possible. Therefore, we wish to minimize the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((x_i - x_0)^2 + (y_i - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the

resulting nonlinear system of three equations in the three unknowns \(x_0, y_0,\) and \(R\).

> eq1 := diff(F(x0, y0, R), x0) = 0;

\[
eq 1 := 2 ((50 - x_0)^2 + (1.2 - y_0)^2 - R^2) (-100 + 2 x_0)
+ 2 ((100 - x_0)^2 + (5 - y_0)^2 - R^2) (-200 + 2 x_0)
+ 2 ((150 - x_0)^2 + (12 - y_0)^2 - R^2) (-300 + 2 x_0)
+ 2 ((200 - x_0)^2 + (23.7 - y_0)^2 - R^2) (-400 + 2 x_0)
+ 2 ((250 - x_0)^2 + (40.5 - y_0)^2 - R^2) (-500 + 2 x_0) = 0
\]

> eq2 := diff(F(x0, y0, R), y0) = 0;

\[
eq 2 := 2 ((50 - x_0)^2 + (1.2 - y_0)^2 - R^2) (-2.4 + 2 y_0)
+ 2 ((100 - x_0)^2 + (5 - y_0)^2 - R^2) (-10 + 2 y_0)
+ 2 ((150 - x_0)^2 + (12 - y_0)^2 - R^2) (-24 + 2 y_0)
+ 2 ((200 - x_0)^2 + (23.7 - y_0)^2 - R^2) (-47.4 + 2 y_0)
+ 2 ((250 - x_0)^2 + (40.5 - y_0)^2 - R^2) (-81.0 + 2 y_0) = 0
\]

> eq3 := diff(F(x0, y0, R), R) = 0;

\[
eq 3 := -4 ((50 - x_0)^2 + (1.2 - y_0)^2 - R^2) \cdot R - 4 ((100 - x_0)^2 + (5 - y_0)^2 - R^2) \cdot R
- 4 ((150 - x_0)^2 + (12 - y_0)^2 - R^2) \cdot R - 4 ((200 - x_0)^2 + (23.7 - y_0)^2 - R^2) \cdot R
- 4 ((250 - x_0)^2 + (40.5 - y_0)^2 - R^2) \cdot R = 0
\]

> sol := solve({eq1, eq2, eq3}, {x0, y0, R});
\begin{verbatim}
sol := \{ R = 0, x0 = 149.9989879 - 122.5279008 I, y0 = 14.93667475 - 23.91733852 I \}.
| R = 0, x0 = 149.9989879 + 122.5279008 I, y0 = 14.93667475 + 23.91733852 I \}.
| R = 0, y0 = 19.53195143, x0 = 150.0088036 \},
| R = 0, x0 = 151.1632857 - 13.83915037 I, y0 = 15.18174545 + 71.02797024 I \}.
| R = 0, y0 = 15.18174545 - 71.02797024 I, x0 = 151.1632857 + 13.83915037 I \}.
| x0 = 35.76359666, y0 = 606.6326482, R = -605.4196806 \},
| x0 = 35.76359666, y0 = 606.6326482, R = 605.4196806 \}
> circ := (x-x0)^2+(y-y0)^2=R^2;
> with(plots);
[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
conourplot, contourplot3d, cooardplot, coordplot3d, cylinderplot, densityplot, display, display3d,
fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedra-supported,
polyhedraplot, replot, rootlocus, semilogplot, setoptions, setoptions3d, spacecurve,
sparsematrixplot, sphereplot, surfdata, textplot, textplot3d, tubeplot]
> p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);
> with(stats[statplots]);
[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,
yxexchange, zscale, zshift]
> p2 := scatterplot(xx, yy, color=black);
> display([p1, p2], scaling=constrained);
\end{verbatim}
This worksheet is to find a circle which best fits a list of data points. The \(x\) coordinates are given as a list \(xx\), and the corresponding \(y\) coordinates are given as a list \(yy\). The number of data points is \(n\). The data \(T\) entered was labelled. 

\[
\text{restart;}
\]
\[
xx := [50, 150, 250];
\]
\[
yy := [1.2, 12, 40, 5];
\]
\[
n := 3;
\]

If we find a circle \((x - x_0)^2 + (y - y_0)^2 = R^2\) which passes exactly though the given points \([x_i, y_i]\), \(i = 1 \ldots n\), then the \(n\) quantities \((x_i - x_0)^2 + (y_i - y_0)^2 - R^2\) will all be zero. This is too much to hope for, but we can seek three numbers \(x_0, y_0, R\), so that the sum of the squares of these quantities is as small as possible. Therefore, we wish to find the following function \(F\).

\[
F := (x_0, y_0, R) \rightarrow \sum_{i=1}^{n} ((xx[i] - x_0)^2 + (yy[i] - y_0)^2 - R^2)^2
\]

We seek a critical point for the function \(F\) by setting its partial derivatives equal to 0, and solving the resulting nonlinear system of three equations in the three unknowns \(x_0, y_0, R\).

\[
eq 1 := \text{diff}(F(x_0, y_0, R), x_0) = 0;
\]
\[
eq 2 := \text{diff}(F(x_0, y_0, R), y_0) = 0;
\]
\[
eq 3 := \text{diff}(F(x_0, y_0, R), R) = 0;
\]

\[
\text{sol} := \text{solve}([\text{eq1}, \text{eq2}, \text{eq3}], [x_0, y_0, R]);
\]

\[
sol := \{R = 0, y_0 = 16.42580731 \pm 27.87922603 l, x_0 = 150.0009353 \pm 141.4780218 l\}.
\]

\[
\{R = 0, y_0 = 16.42580731 + 27.87922603 l, x_0 = 150.0009353 + 141.4780218 l\},
\]

\[
\{R = 0, y_0 = 16.42580731 - 27.87922603 l, x_0 = 150.0009353 - 141.4780218 l\},
\]

\[
\{R = 0, y_0 = 16.42580731, x_0 = 20.81781352\}.
\]

\[
\{R = 0, y_0 = 16.66120692 + 81.98486896 l, x_0 = 151.1156278 - 16.13553059 l\},
\]

\[
\{R = 0, y_0 = 16.66120692 - 81.98486896 l, x_0 = 151.1156278 + 16.13553059 l\},
\]

\[
\{x_0 = 35.56594915, y_0 = 603.2115819, R = -602.1845951\}.
\]
\{ x0 = 35.56594915, y0 = 603.2115819, R = 602.1845951 \}

\begin{verbatim}
> circ := (x-x0)^2 + (y-y0)^2 = R^2;

> with(plots);

[animate, animate3d, animatecurve, changecoords, complexplot, complexplot3d, conformal,
    contourplot, contourplot3d, coordplot, coordplot3d, cylinderplot, densityplot, display, display3d,
    fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, inequal, listcontplot,
    listcontplot3d, listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, odeplot, pareto,
    pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, polyhedral_supported,
    polyhedrplot, replot, root locus, semilogplot, setoptions, setoptions3d, spacecurve,
    sparsematrixplot, sphereplot, surfdata, texplot, texpplot3d, tubeplot]

> p1 := implicitplot(subs(sol[7], circ), x=0..400, y=-300..300, numpoints=4000);

> with(stats[statplots]);

[boxplot, histogram, scatterplot, xscale, xshift, xyexchange, xexchange, yscale, yshift,
    yexchange, zscale, zshift]

> p2 := scatterplot(xx, yy, color=black);

> display([p1, p2], scaling=constrained);
\end{verbatim}
This worksheet will find circles through consecutive sets of three points. The way to use it is to enter the list of x coordinates as xx and the list of y coordinates as yy. The number of points in the list is n.

The index of the loop is i, which refers to the index of the first of the three points in the list. The output of the loop at each step is i and then the center and radius of the circle (unfortunately, not always in the same order). There is always a spurious solution corresponding to a negative R. If the three points are collinear, then no solution can be found, so the loop simply outputs the value of i. The data I entered was labelled, "Futlock III, B8E".

```math
\text{restart;}
xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 276];
xx := [0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 210, 220, 230, 240, 250, 260, 270, 276];

yy := [3.8, 2.9, 2.5, 2.1, 8.1, 1.2, 0.5, 1.6, 2.3, 7.5, 6.3, 8.9, 10.7, 12, 14.3, 16.8, 19, 21.5, 23.7, 27.2, 30.5, 33.5, 37, 40.5, 44, 47.5, 50];

yy := [3.8, 2.9, 2.5, 2.1, 8.1, 1.2, 0.5, 1.6, 2.3, 7.5, 6.3, 8.9, 10.7, 12, 14.3, 16.8, 19, 21.5, 23.7, 27.2, 30.5, 33.5, 37, 40.5, 44, 47.5, 50];

\text{cir} := \{(x, y) \rightarrow (x-x_0)^2 + (y-y_0)^2 = R^2\};
\text{cir} := \{(x, y) \rightarrow (x-x_0)^2 + (y-y_0)^2 = R^2\};

n := 29;
n := 29;

\text{for} i \text{ from 1 to } n-2 \text{ do}
\text{for} i \text{ from 1 to } n-2 \text{ do}

\text{eq1} := \text{cir}(\text{xx}[i], \text{yy}[i]);
\text{eq1} := \text{cir}(\text{xx}[i], \text{yy}[i]);

\text{eq2} := \text{cir}(\text{xx}[i+1], \text{yy}[i+1]);
\text{eq2} := \text{cir}(\text{xx}[i+1], \text{yy}[i+1]);

\text{eq3} := \text{cir}(\text{xx}[i+2], \text{yy}[i+2]);
\text{eq3} := \text{cir}(\text{xx}[i+2], \text{yy}[i+2]);

\text{print}(i, \text{solve}([\text{eq1, eq2, eq3}], [x_0, y_0, R]));
\text{print}(i, \text{solve}([\text{eq1, eq2, eq3}], [x_0, y_0, R]));

end:
end:

1. \{y_0 = 203.8700000, x_0 = 23.046680000, R = 201.3930483\},
\{y_0 = 203.8700000, x_0 = 23.046680000, R = 201.3930483\},

2. \{y_0 = -999.5500000, x_0 = -25.09000000, R = 1003.063961\},
\{y_0 = -999.5500000, x_0 = -25.09000000, R = 1003.063961\},

3. \{R = -334.0208832, y_0 = 335.8166667, x_0 = 41.67833333\},
\{R = -334.0208832, y_0 = 335.8166667, x_0 = 41.67833333\},

4. \{x_0 = 29.988000000, y_0 = -248.7000000, R = 250.70000003\},
\{x_0 = 29.988000000, y_0 = -248.7000000, R = 250.70000003\},

5. \{x_0 = -15.273000000, y_0 = -1003.0500000, R = 1006.369031\},
\{x_0 = -15.273000000, y_0 = -1003.0500000, R = 1006.369031\},

6. \{x_0 = 58.89744444, y_0 = 56.52777778, R = -56.03862517\},
\{x_0 = 58.89744444, y_0 = 56.52777778, R = -56.03862517\},

7. \{y_0 = -142.2357143, x_0 = 80.76142857, R = -144.2377241\},
\{y_0 = -142.2357143, x_0 = 80.76142857, R = -144.2377241\},
```
8. \{ x_0 = 71.86815385, y_0 = 80.09615385, R = -78.51838108 \}
   \{ x_0 = 71.86815385, y_0 = 80.09615385, R = 78.51838108 \}
9. \{ x_0 = 128.32875000, y_0 = -252.02500000, R = -258.5814547 \}
   \{ x_0 = 128.32875000, y_0 = -252.02500000, R = 258.5814547 \}

10. \{ R = -258.5814547, x_0 = 71.67125000, y_0 = 262.02500000 \}
    \{ R = 258.5814547, x_0 = 71.67125000, y_0 = 262.02500000 \}
11. \{ x_0 = 139.61357143, R = -146.9505136, y_0 = -137.6357143 \}
    \{ R = 146.9505136, x_0 = 139.61357143, y_0 = -137.6357143 \}
12. \{ x_0 = 110.3864286, y_0 = 154.6357143, R = -146.9505136 \}
    \{ x_0 = 110.3864286, y_0 = 154.6357143, R = 146.9505136 \}
13. \{ R = -258.5814547, x_0 = 178.3287500, y_0 = -245.02500000 \}
    \{ R = 258.5814547, x_0 = 178.3287500, y_0 = -245.02500000 \}
14. \{ y_0 = 115.4900000, x_0 = 131.4618000, R = -105.1372672 \}
    \{ y_0 = 115.4900000, x_0 = 131.4618000, R = 105.1372672 \}
15. \{ x_0 = 33.10000000, y_0 = 543.15000000, R = -543.8620528 \}
    \{ x_0 = 33.10000000, y_0 = 543.15000000, R = 543.8620528 \}
16. \{ R = -361.3926009, x_0 = 252.6416667, y_0 = -335.0166667 \}
    \{ R = 361.3926009, x_0 = 252.6416667, y_0 = -335.0166667 \}
17. \{ x_0 = 97.3583333, y_0 = 370.8166667, R = -361.3926009 \}
    \{ x_0 = 97.3583333, y_0 = 370.8166667, R = 361.3926009 \}
18. \{ x_0 = 272.6416667, y_0 = -330.3166667, R = -361.3926009 \}
    \{ x_0 = 272.6416667, y_0 = -330.3166667, R = 361.3926009 \}
19. \{ x_0 = 176.3888462, y_0 = 107.1961538, R = -86.77035377 \}
    \{ x_0 = 176.3888462, y_0 = 107.1961538, R = 86.77035377 \}
20. \{ R = -589.2012014, x_0 = 399.6350000, y_0 = -530.6500000 \}
    \{ R = 589.2012014, x_0 = 399.6350000, y_0 = -530.6500000 \}
21. \{ x_0 = 335.3950000, R = -384.2213420, y_0 = -335.9833333 \}
    \{ R = 384.2213420, x_0 = 335.3950000, y_0 = -335.9833333 \}
22. \{ x_0 = 158.1750000, y_0 = 254.7500000, R = -232.6164077 \}
    \{ R = 232.6164077, x_0 = 158.1750000, y_0 = 254.7500000 \}

24

25

26

27. \{ y_0 = 184.5000000, x_0 = 216.4375000, R = -147.0984072 \}.
\{ y_0 = 184.5000000, x_0 = 216.4375000, R = 147.0984072 \}